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Motion About the Stable Libration Points in the Linearized, Restricted Three-Body Problem

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FIGURES

3   Typical trajectories for which $C = 0$ (continued)

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*Don Mittleman**

1.0 SUMMARY

Starting from the well-known differential equations of motion for a point particle in the neighborhood of a triangle libration point (L_4 or L_5) in the linearized, restricted problem of three bodies in the plane, and after obtaining the finite equations of motion, three integral invariants of the motion are derived. The Jacobi integral is expressed linearly in terms of two of these. The trajectories for varied initial conditions are drawn; the motion of a point particle, as it moves along its trajectory, can be visualized in elementary geometric terms. Each trajectory has an exterior bounding curve; its equation is found. An approximation to this boundary curve, a well-defined ellipse, had been known; the two curves are compared graphically. For certain initial conditions, there is an interior region from which the trajectory is excluded; the equation of the boundary of this region is found.

2.0 INTRODUCTION

The motion of a particle in the neighborhood of a triangular libration point in the linearized, restricted problem of three bodies in the plane is investigated. In section 3, the linearized equations of motion are given; in section 4, three invariants of the motion are determined; and, in section 5, the trajectories for several initial conditions are discussed. All trajectories are bounded; thus, in section 6, the equations for the exterior bounding curve are derived and, where appropriate, the equations for the interior boundary curves are given.

3.0 THE EQUATIONS OF MOTION

In section 3.1, the differential equations of motion for a particle are derived. The Jacobi integral and the

Jacobi ellipse are obtained. Generic solutions of the differential equations are given in section 3.2 and their dependency on initial conditions is shown in section 3.3. Various combinations of the constants of the system are interrelated; these relationships are given in section 3.4.

3.1 The Differential Equations

The following discussion parallels closely that given in reference 1. Two bodies of finite mass, shown as the Earth (E) and the Moon (M) in the following sketch, are assumed to rotate about an axis ζ_1 through their center of mass in circular orbits lying in a plane π that is orthogonal to ζ_1. A rotating rectangular coordinate system (ξ_1, η_1) lies in π; the Earth and Moon have the fixed coordinates ($x_1, 0, 0$) and ($x_2, 0, 0$), respectively, and their center of mass is at ($0, 0, 0$). The distance between E and M is chosen to be equal to 1; the mass of M is taken as $\mu \leq 0.5$, that of E as $1 - \mu$. The choice for the unit of time is such that the gravitational constant $k^2 = 1$.[1]
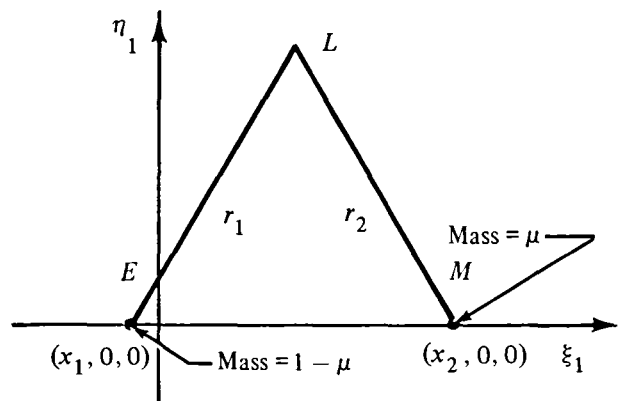
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In the three-dimensional (ξ_1,η_1,ζ_1) coordinate system, the differential equations of motion for a point particle are

$$\frac{d^2\xi_1}{dt^2} - 2\frac{d\eta_1}{dt} = \xi_1 - \frac{(1-\mu)(\xi_1 - x_1)}{r_1{}^3} - \frac{\mu(\xi_1 - x_2)}{r_2{}^3}$$

$$\frac{d^2\eta_1}{dt^2} + 2\frac{d\xi_1}{dt} = \eta_1 - \frac{(1-\mu)\eta_1}{r_1{}^3} - \frac{\mu\eta_1}{r_2{}^3}$$

$$\frac{d^2\zeta_1}{dt^2} = -\frac{(1-\mu)\zeta_1}{r_1{}^3} - \frac{\mu\zeta_1}{r_2{}^3}$$

where $r_1{}^2 = (\xi_1 - x_1)^2 + \eta_1{}^2 + \zeta_1{}^2$ and $r_2{}^2 = (\xi_1 - x_2)^2 + \eta_1{}^2 + \zeta_1{}^2$. If the potential function

$$U(\xi_1,\eta_1,\zeta_1) = \frac{1}{2}\left(\xi_1{}^2 + \eta_1{}^2\right) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2}$$

is introduced, the equations of motion become

$$\frac{d^2\xi_1}{dt^2} - 2\frac{d\eta_1}{dt} = \frac{\partial U}{\partial \xi_1}$$

$$\frac{d^2\eta_1}{dt^2} + 2\frac{d\xi_1}{dt} = \frac{\partial U}{\partial \eta_1}$$

$$\frac{d^2\zeta_1}{dt^2} = \frac{\partial U}{\partial \zeta_1}$$

If a (ξ_2,η_2,ζ_2) coordinate system is defined with the origin at L, a triangular libration point, and with the axes parallel to the (ξ_1,η_1,ζ_1) axes, the *linearized* equations of motion are

$$\frac{d^2\xi_2}{dt^2} - 2\frac{d\eta_2}{dt} = \frac{3}{4}\xi_2 + \frac{3\sqrt{3}}{4}(1 - 2\mu)\eta_2 \qquad (1a)$$

$$\frac{d^2\eta_2}{dt^2} + 2\frac{d\xi_2}{dt} = \frac{3\sqrt{3}}{4}(1 - 2\mu)\xi_2 + \frac{9}{4}\eta_2 \qquad (1b)$$

$$\frac{d^2\zeta_2}{dt^2} = -\zeta_2 \qquad (1c)$$

Equation (1c), being independent of equations (1a) and (1b), shows that the motion parallel to the ζ_2 axis is periodic with period 2π. No more will be said about this; the remaining discussion concentrates on the solutions of the first two equations.

If equation (1a) is multiplied by $d\xi_2/dt$ and equation (1b) by $d\eta_2/dt$ and the results added, the following is obtained.

$$\frac{d\xi_2}{dt}\frac{d^2\xi_2}{dt^2} + \frac{d\eta_2}{dt}\frac{d^2\eta_2}{dt^2} =$$

$$\frac{3}{4}\xi_2\frac{d\xi_2}{dt} + \frac{3\sqrt{3}}{4}(1 - 2\mu)\left(\eta_2\frac{d\xi_2}{dt} + \xi_2\frac{d\eta_2}{dt}\right) + \frac{9}{4}\eta_2\frac{d\eta_2}{dt}$$

This equation can be integrated and yields the Jacobi integral

$$\frac{1}{2}\left[\left(\frac{d\xi_2}{dt}\right)^2 + \left(\frac{d\eta_2}{dt}\right)^2\right] = \frac{3}{4}\frac{\xi_2{}^2}{2} + \frac{3\sqrt{3}}{4}(1 - 2\mu)\xi_2\eta_2$$

$$+ \frac{9}{4}\frac{\eta_2{}^2}{2} + \frac{C}{2} \qquad (2)$$

where C is a constant to be determined from the initial conditions. In particular, three different values for C are distinguished: -1, 0, and $+1$. Since equations (1a) and (1b) are linear in ξ_2 and η_2 and equation (2) is a homogeneous function of ξ_2, η_2, $d\xi_2/dt$, and $d\eta_2/dt$, any other value of C can be reduced to one of these three by properly scaling the (ξ_2, η_2) variables.

3.1.1 Rest trajectories.—Those trajectories for which the initial velocity is zero will be called "rest trajectories." For these, $C = -1$ and the initial positions lie on the conic

$$\frac{3}{8}\xi_2^2 + \frac{3\sqrt{3}}{4}(1 - 2\mu)\xi_2\eta_2 + \frac{9}{8}\eta_2^2 = \frac{1}{2}$$

If $0 < \mu < 1$, the discriminant of the quadratic is

$$D = \left[\frac{3\sqrt{3}}{4}(1 - 2\mu)\right]^2 - 4\left(\frac{27}{64}\right) < 0$$

and the conic is an ellipse which shall be called the Jacobi ellipse.

3.1.2 Simplified form of the equations of motion.— To simplify subsequent discussions, an (x,y) coordinate system is introduced. The origin is taken at L, which is also the center of the Jacobi ellipse; the x axis coincides in direction with the major axis of the ellipse and the y axis with the minor axis. This is effectively a rotation of axes with respect to the (ξ_2, η_2) axes through an angle θ, where $\theta = 0.5 \arctan [-\sqrt{3}(1 - 2\mu)]$ radians.

In the (x,y) coordinate system, the linearized equations of motion (equations (1a) and (1b)) become

$$\ddot{x} - 2\dot{y} = \omega_1^2 x \qquad (3a)$$

$$\ddot{y} + 2\dot{x} = \omega_2^2 y \qquad (3b)$$

where

$$\omega_1^2 = \frac{3}{2}\left(1 - \sqrt{1 - 3\mu + 3\mu^2}\right)$$

$$\omega_2^2 = \frac{3}{2}\left(1 + \sqrt{1 - 3\mu + 3\mu^2}\right)$$

and the dots indicate differentiation with respect to time.

In this coordinate system, the Jacobi integral is

$$\dot{x}^2 + \dot{y}^2 = \omega_1^2 x^2 + \omega_2^2 y^2 + C \qquad (3c)$$

and the Jacobi ellipse is

$$\omega_1^2 x^2 + \omega_2^2 y^2 = 1 \qquad (3d)$$

3.2 Solutions of the Differential Equations

The solutions to equations (3a) and (3b) are of the form

$$x = K_1 \cos \lambda t + K_2 \sin \lambda t$$

$$y = L_1 \cos \lambda t + L_2 \sin \lambda t$$

Nonzero solutions can be obtained if λ satisfies the characteristic equation:

$$\lambda^4 - \lambda^2 + \frac{27}{4}\mu(1 - \mu) = 0;$$

$$\lambda_1^2 = \frac{1 + \sqrt{1 - 27\mu(1 - \mu)}}{2}$$

$$\lambda_2^2 = \frac{1 - \sqrt{1 - 27\mu(1 - \mu)}}{2}$$

$$\lambda_3 = -\lambda_1$$

$$\lambda_4 = -\lambda_2$$

Only the stable solutions are considered; i.e., those for which the λ are real and not equal to zero. Acceptable values of μ are those for which $1 - 27\mu(1 - \mu) > 0$ or $\mu < 0.0385207\ldots$. For the Earth-Moon system, $\mu = 1/82.45 \approx 0.0121285\ldots$ and the linear theory may be used to describe the motion of a satellite near the libration point.

The solutions to equations (3a) and (3b) are among the functions

$$x = K_1 \cos \lambda_1 t + K_2 \sin \lambda_1 t + K_3 \cos \lambda_2 t + K_4 \sin \lambda_2 t$$

$$y = L_1 \cos \lambda_1 t + L_2 \sin \lambda_1 t + L_3 \cos \lambda_2 t + L_4 \sin \lambda_2 t$$

where $K_1, \ldots, L_4$ are arbitrary constants. If these trial solutions are to satisfy equations (3a) and (3b), the coefficients $K_1, \ldots, L_4$ are related. The solutions to equations (3a) and (3b) can be represented by

$$x = \left(K_1 \cos \lambda_1 t + K_2 \sin \lambda_1 t \right)$$
$$+ \left(K_3 \cos \lambda_2 t + K_4 \sin \lambda_2 t \right) \qquad (4a)$$

$$y = M_1 \left(K_2 \cos \lambda_1 t - K_1 \sin \lambda_1 t \right)$$
$$+ M_2 \left(K_4 \cos \lambda_2 t - K_3 \sin \lambda_2 t \right) \qquad (4b)$$

where

$$M_1 = \frac{\omega_1{}^2 + \lambda_1{}^2}{2\lambda_1}$$

and

$$M_2 = \frac{\omega_1{}^2 + \lambda_2{}^2}{2\lambda_2}$$

3.3 Initial Conditions

The dependency of K_1, K_2, K_3, and K_4 on the initial conditions follows. If, at time $t = 0$, $x = x_o$, $y = y_o$, $\dot{x} = \dot{x}_o$, and $\dot{y} = \dot{y}_o$, then

$$K_1 = \frac{\left(\omega_1{}^2 + \lambda_2{}^2 \right) x_o + 2\dot{y}_o}{\lambda_2{}^2 - \lambda_1{}^2} \qquad (5a)$$

$$K_2 = \frac{\lambda_1}{\omega_1{}^2} \left[\frac{2\lambda_2{}^2 y_o - \left(\omega_1{}^2 + \lambda_2{}^2 \right) \dot{x}_o}{\lambda_2{}^2 - \lambda_1{}^2} \right] \qquad (5b)$$

$$K_3 = - \frac{\left(\omega_1{}^2 + \lambda_1{}^2 \right) x_o + 2\dot{y}_o}{\lambda_2{}^2 - \lambda_1{}^2} \qquad (5c)$$

$$K_4 = \frac{\lambda_2}{\omega_1{}^2} \left[\frac{\left(\omega_1{}^2 + \lambda_1{}^2 \right) \dot{x}_o - 2\lambda_1{}^2 y_o}{\lambda_2{}^2 - \lambda_1{}^2} \right] \qquad (5d)$$

The determination of K_i ($i = 1, 2, 3, 4$) is obtained from equations (4a) and (4b) in a straightforward, if tedious, calculation.

3.4 Some Useful Relationships

By direct computation, it can be shown that

$$\omega_1{}^2 + \omega_2{}^2 = 3$$

$$\lambda_1{}^2 + \lambda_2{}^2 = 1$$

$$\omega_1 \omega_2 = \lambda_1 \lambda_2$$

$$M_1 = \frac{\omega_1{}^2 + \lambda_1{}^2}{2\lambda_1} = \frac{2\lambda_1}{\omega_2{}^2 + \lambda_1{}^2}$$

$$M_2 = \frac{\omega_1{}^2 + \lambda_2{}^2}{2\lambda_2} = \frac{2\lambda_2}{\omega_2{}^2 + \lambda_2{}^2}$$

and

$$M_1 M_2 = \frac{\omega_1}{\omega_2}$$

Alternately, if one starts with the equation

$$\ddot{x} - 2\dot{y} = \omega_1^2 x \qquad (3a)$$

$$\ddot{y} + 2\dot{x} = \omega_2^2 y \qquad (3b)$$

and seeks a solution of the form

$$x = K_1 \cos \lambda t + K_2 \sin \lambda t$$

$$y = L_1 \cos \lambda t + L_2 \sin \lambda t$$

the characteristic equation for λ is

$$\lambda^4 + \left(\omega_1^2 + \omega_2^2 - 4\right)\lambda^2 + \omega_1^2 \omega_2^2 = 0 \qquad (6)$$

From this, it is inferred that

$$\lambda_1^2 + \lambda_2^2 = -\left(\omega_1^2 + \omega_2^2 - 4\right)$$

or

$$\lambda_1^2 + \lambda_2^2 + \omega_1^2 + \omega_2^2 = 4$$

and, since $\omega_1^2 + \omega_2^2 = 3$, $\lambda_1^2 + \lambda_2^2 = 1$. From $\lambda_1^2 \lambda_2^2 = \omega_1^2 \omega_2^2$, it is inferred that $\lambda_1 \lambda_2 = \omega_1 \omega_2$, disregarding the negative value for the square root.

Since λ_i $(i = 1, 2)$ is a root of the characteristic equation (6), which can be written in the form

$$\frac{\omega_1^2 + \lambda^2}{2\lambda} = \frac{2\lambda}{\omega_2^2 + \lambda^2}$$

the alternate forms for M_1 and M_2 become apparent. Finally, from

$$M_1 = \frac{2\lambda_1}{\omega_2^2 + \lambda_1^2}$$

$$= \frac{2\left(\dfrac{\omega_1 \omega_2}{\lambda_2}\right)}{\omega_2^2 + \lambda_1^2}$$

$$= \frac{2\omega_1 \omega_2}{\lambda_2 \omega_2^2\left[1 + \left(\dfrac{\lambda_1}{\omega_2}\right)^2\right]}$$

$$= \frac{2\omega_1}{\lambda_2 \omega_2\left[1 + \left(\dfrac{\omega_1}{\lambda_2}\right)^2\right]}$$

$$= \frac{2\omega_1 \lambda_2}{\omega_2\left(\lambda_2^2 + \omega_1^2\right)}$$

$$= \frac{\omega_1}{\omega_2}\left(\frac{2\lambda_2}{\omega_1^2 + \lambda_2^2}\right)$$

$$= \frac{\omega_1}{\omega_2}\frac{1}{M_2}$$

$$M_1 M_2 = \frac{\omega_1}{\omega_2}$$

4.0 INVARIANTS OF THE MOTION

In section 4.1, two invariants of the motion, J_1 and J_2, are derived. A third invariant relationship is given in section 4.2, where it is also shown that these three invariants can be used to distinguish between different trajectories or to identify a "second" trajectory as a continuation of an earlier identified one. A linear relation exists between J_1^2, J_2^2 and the Jacobi constant; this is derived in section 4.3.

4.1 The J_1 and J_2 Invariants

Suppose a particle P is observed at $t = t_0$ at position (x_o, y_o) with velocity $(\dot{x}_o, \dot{y}_o)$. Its trajectory can be determined. At some later time t_1, the particle P will be at the point $[x(t_1), y(t_1)] = (x_1, y_1)$ with velocity $[\dot{x}(t_1), \dot{y}(t_1)] = (\dot{x}_1, \dot{y}_1)$. However, between $t = t_0$ and $t = t_1$, you lose track of the particle. Is it possible to decide if the particle P observed at time $t = t_0$ and the particle observed at $t = t_1$ are the same? The following discussion partly answers this question.

Having observed the particle initially at (x_o, y_o) with "initial" velocity $(\dot{x}_o, \dot{y}_o)$, a unique trajectory and, in particular, the numbers K_1, K_2, K_3, and K_4 are determined. At the later time t_1, a particle is observed at the point (x_1, y_1) with velocity $(\dot{x}_1, \dot{y}_1)$. For these "initial" conditions, a unique trajectory and, in particular, four numbers K_1^*, K_2^*, K_3^*, and K_4^* are determined. The question may then be framed: Is this second trajectory merely a continuation of the first or is a different particle being observed?

Using t to represent time for the "first" particle (so that at t_0, $t = 0$), its equations of motion are

$$x(t) = K_1 \cos \lambda_1 t + K_2 \sin \lambda_1 t$$
$$+ K_3 \cos \lambda_2 t + K_4 \sin \lambda_2 t \qquad (7a)$$

$$y(t) = M_1 \left(K_2 \cos \lambda_1 t - K_1 \sin \lambda_1 t \right)$$
$$+ M_2 \left(K_4 \cos \lambda_2 t - K_3 \sin \lambda_2 t \right) \qquad (7b)$$

Using τ to represent time for the "second" particle (so that at t_1, $\tau = 0$), its equations of motion are

$$x(\tau) = K_1^* \cos \lambda_1 \tau + K_2^* \sin \lambda_1 \tau$$
$$+ K_3^* \cos \lambda_2 \tau + K_4^* \sin \lambda_2 \tau \qquad (8a)$$

$$y(\tau) = M_1 \left(K_2^* \cos \lambda_1 \tau - K_1^* \sin \lambda_1 \tau \right)$$
$$+ M_2 \left(K_4^* \cos \lambda_2 \tau - K_3^* \sin \lambda_2 \tau \right) \qquad (8b)$$

The relation between t and τ is $t = \tau + t_1$.

Equations (7a) and (7b) for the original trajectory $[x(t), y(t)]$ when viewed in the time frame τ become

$$x(\tau + t_1) = K_1 \cos \lambda_1 (\tau + t_1) + K_2 \sin \lambda_1 (\tau + t_1)$$
$$+ K_3 \cos \lambda_2 (\tau + t_1) + K_4 \sin \lambda_2 (\tau + t_1)$$
$$(9a)$$

$$y(\tau + t_1) = M_1 \left[K_2 \cos \lambda_1 (\tau + t_1) - K_1 \sin \lambda_1 (\tau + t_1) \right]$$
$$+ M_2 \left[K_4 \cos \lambda_2 (\tau + t_1) - K_3 \sin \lambda_2 (\tau + t_1) \right]$$
$$(9b)$$

Using the addition formulas for $\sin \lambda(\tau + t_1)$ and $\cos \lambda(\tau + t_1)$ in equations (9a) and (9b) and equating the coefficients of $\cos \lambda_1 \tau$, $\sin \lambda_1 \tau$, $\cos \lambda_2 \tau$, and $\sin \lambda_2 \tau$ to the corresponding quantities in equations (8a) and (8b),[2] it is found that

$$K_1^* = K_1 \cos \lambda_1 t_1 + K_2 \sin \lambda_1 t_1 \qquad (10a)$$

$$K_2^* = K_2 \cos \lambda_1 t_1 - K_1 \sin \lambda_1 t_1 \qquad (10b)$$

$$K_3^* = K_3 \cos \lambda_2 t_1 + K_4 \sin \lambda_2 t_1 \qquad (10c)$$

$$K_4^* = K_4 \cos \lambda_2 t_1 - K_3 \sin \lambda_2 t_1 \qquad (10d)$$

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[2]A sufficient condition to permit the equating of the corresponding coefficients is that λ_1 and λ_2 be incommensurate. This is assumed here and throughout the remainder of this paper.

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However, equation (10) implies that

$$K_1^{*2} + K_2^{*2} = K_1^2 + K_2^2 \qquad \text{(11a)}$$

and

$$K_3^{*2} + K_4^{*2} = K_3^2 + K_4^2 \qquad \text{(11b)}$$

Therefore, $K_1^2 + K_2^2$ and $K_3^2 + K_4^2$ are invariants of the trajectory. The notations $J_1^2 = K_1^2 + K_2^2$ and $J_2^2 = K_3^2 + K_4^2$ shall be used.

4.2 The Third Invariant

It has been shown that the quantities J_1 and J_2 are invariants of the trajectory. The question of additional invariants is pursued.

From equations (10a) to (10d), it is found that

$$\cos \lambda_1 t_1 = \frac{K_1 K_1^* + K_2 K_2^*}{J_1^2}$$

$$\sin \lambda_1 t_1 = \frac{K_2 K_1^* - K_1 K_2^*}{J_1^2}$$

$$\cos \lambda_2 t_1 = \frac{K_3 K_3^* + K_4 K_4^*}{J_2^2}$$

$$\sin \lambda_2 t_1 = \frac{K_4 K_3^* - K_3 K_4^*}{J_2^2}$$

(Note that, on the right sides of these equations, only K_1^*, K_2^*, K_3^*, and K_4^* depend on t_1; the other symbols do not.)

Four angles ϕ_1, ϕ_1^*, ϕ_2, and ϕ_2^* are defined by the following relations:

$$\cos \phi_1 = \frac{K_1}{J_1} \qquad \sin \phi_1 = \frac{K_2}{J_1}$$

$$\cos \phi_1^* = \frac{K_1^*}{J_1^*} \qquad \sin \phi_1^* = \frac{K_2^*}{J_1^*}$$

$$\cos \phi_2 = \frac{K_3}{J_2} \qquad \sin \phi_2 = \frac{K_4}{J_2}$$

$$\cos \phi_2^* = \frac{K_3^*}{J_2^*} \qquad \sin \phi_2^* = \frac{K_4^*}{J_2^*}$$

(Remember that $J_1^* = J_1$ and $J_2^* = J_2$. Also, only ϕ_1^* and ϕ_2^* are functions of t_1.) Using this notation,

$$\cos \left(\lambda_1 t_1 \right) = \cos \phi_1 \cos \phi_1^* + \sin \phi_1 \sin \phi_1^*$$

$$= \cos \left(\phi_1 - \phi_1^* \right)$$

$$\sin \left(\lambda_1 t_1 \right) = \sin \phi_1 \cos \phi_1^* - \cos \phi_1 \sin \phi_1^*$$

$$= \sin \left(\phi_1 - \phi_1^* \right)$$

Therefore, there exists an integer n such that

$$\lambda_1 t_1 = \phi_1 - \phi_1^* + 2n\pi \qquad \text{(12a)}$$

It is claimed that n is zero. Consider

$$2\pi n = \lambda_1 t_1 - \phi_1 + \phi_1^* \qquad \text{(12b)}$$

The symbol ϕ_1, which depends on the "initial" conditions at t_0, is independent of t_1 and for our purpose is constant. The symbol ϕ_1^* depends on the "initial"

conditions at t_1; ϕ_1^* is a continuous function of t_1. Furthermore, if $t_1 = 0$, $\phi_1^* = \phi_1$. Therefore, the right side of equation (12b) is zero for $t_1 = 0$. Thus, $n = 0$ when $t_1 = 0$. But n is an integer and the right side of equation (12b) is a continuous function of t_1. This implies that $n \equiv 0$ and equation (12a) becomes

$$\lambda_1 t_1 = \phi_1 - \phi_1^* \qquad (12c)$$

A similar argument shows that

$$\lambda_2 t_1 = \phi_2 - \phi_2^* \qquad (12d)$$

Eliminating t_1,

$$\lambda_2 \phi_1 - \lambda_1 \phi_2 = \lambda_2 \phi_1^* - \lambda_1 \phi_2^* \qquad (13)$$

The symbol used shall be $J_3 = \lambda_2 \phi_1 - \lambda_1 \phi_2$.

Equations (11) and (13) show that if the same trajectory is observed at two different times, the quantities J_1, J_2, and J_3 are invariants. Conversely, if two trajectories are given for which the J_1, J_2, and J_3 of the first are equal, respectively, to the J_1^*, J_2^*, and J_3^* of the second, then the two trajectories are *not* distinct; one is a continuation at some later time of the other. Furthermore, this later time can be determined from either equation (12c) or equation (12d).

4.3 On the Relation Between J_1, J_2, and C, the Jacobi Constant

The Jacobi integral, equation (3c), is valid for all values of the time t and in particular for $t = 0$. The following relationships between x_o, y_o, $\dot{x}_o$, and $\dot{y}_o$ and K_1, K_2, K_3, and K_4 can be found from equations (4a) and (4b) and their derivatives with respect to time.

$$x_o = K_1 + K_3 \qquad (14a)$$

$$\dot{x}_o = \lambda_1 K_2 + \lambda_2 K_4 \qquad (14b)$$

$$y_o = M_1 K_2 + M_2 K_4 \qquad (14c)$$

$$\dot{y}_o = -\lambda_1 M_1 K_1 - \lambda_2 M_2 K_3 \qquad (14d)$$

When these values are substituted into equation (3c), that equation becomes

$$\left(\lambda_1{}^2 - \omega_2{}^2 M_1{}^2\right) K_2{}^2 + \left(\lambda_1{}^2 M_1{}^2 - \omega_1{}^2\right) K_1{}^2$$

$$+ \left(\lambda_2{}^2 - \omega_2{}^2 M_2{}^2\right) K_4{}^2 + \left(\lambda_2{}^2 M_2{}^2 - \omega_1{}^2\right) K_3{}^2$$

$$+ 2\left(\lambda_1 \lambda_2 - \omega_2{}^2 M_1 M_2\right) K_2 K_4$$

$$+ 2\left(\lambda_1 \lambda_2 M_1 M_2 - \omega_1{}^2\right) K_1 K_3$$

$$= -C \qquad (15)$$

However, since $M_1 M_2 = \omega_1/\omega_2$ and $\lambda_1 \lambda_2 = \omega_1 \omega_2$, the coefficients of $K_2 K_4$ and $K_1 K_3$ are zero.

The coefficients of $K_1{}^2$ and $K_2{}^2$ are shown to be equal by the following. From section 3.4,

$$M_1 = \frac{\omega_1{}^2 + \lambda_1{}^2}{2\lambda_1} = \frac{2\lambda_1}{\omega_2{}^2 + \lambda_1{}^2}$$

so that

$$M_1{}^2 = \frac{\omega_1{}^2 + \lambda_1{}^2}{\omega_2{}^2 + \lambda_1{}^2}$$

$$\omega_2{}^2 M_1{}^2 + \lambda_1{}^2 M_1{}^2 = \omega_1{}^2 + \lambda_1{}^2$$

$$\lambda_1{}^2 M_1{}^2 - \omega_1{}^2 = \lambda_1{}^2 - \omega_2{}^2 M_1{}^2$$

A similar calculation shows that the coefficients of

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K_3^2 and K_4^2 are equal. Thus, equation (15) can be written as

$$\left(\omega_2^2 M_1^2 - \lambda_1^2\right)J_1^2 + \left(\omega_2^2 M_2^2 - \lambda_2^2\right)J_2^2 = C \tag{16}$$

5.0 ILLUSTRATED TRAJECTORIES

Trajectories associated with different initial conditions are explored, in a systematic way, in this section. At a first level of classification, trajectories were differentiated on the basis of the value of the Jacobi constant C. In sections 5.1, 5.2, and 5.3, trajectories for which $C = -1$, $C = +1$, and $C = 0$, respectively, are examined.

5.1 Trajectories for Which $C = -1$

Included among the trajectories for which $C = -1$ are the rest trajectories, those for which the starting position is on the Jacobi ellipse.

$$\omega_1^2 x_o^2 + \omega_2^2 y_o^2 = 1$$

These constitute a one-parameter family of solutions that can be parameterized by $x_o = (1/\omega_1) \cos \gamma$ and $y_o = (1/\omega_2) \sin \gamma$, where γ is the parameter. The values of K_i are

$$K_1 = \frac{\left(\omega_1^2 + \lambda_2^2\right)\cos \gamma}{\omega_1\left(\lambda_2^2 - \lambda_1^2\right)}$$

$$K_2 = \frac{2\lambda_2 \sin \gamma}{\omega_1\left(\lambda_2^2 - \lambda_1^2\right)}$$

$$K_3 = -\frac{\left(\omega_1^2 + \lambda_1^2\right)\cos \gamma}{\omega_1\left(\lambda_2^2 - \lambda_1^2\right)}$$

$$K_4 = -\frac{2\lambda_1 \sin \gamma}{\omega_1\left(\lambda_2^2 - \lambda_1^2\right)}$$

Typical trajectories are drawn for $\gamma = 0°$, $15°$, $30°, \ldots, 345°$ in figure 1. The value of $\mu = 1/82.45$ was chosen, from which λ_1, λ_2, ω_1, ω_2, M_1, and M_2 were calculated.

For the specific initial conditions, K_1, K_2, K_3, K_4, and J_1, J_2 were also calculated. Each graph is scaled so that each mark represents 5 units. The innermost curve (in black) is the Jacobi ellipse; its semi-major axis equals $1/\omega_1$ and its semi-minor axis equals $1/\omega_2$. The trajectory is drawn in red for $0 \leq t \leq 50$.[3] The blue curve will be discussed in section 6.

5.2 Trajectories for Which $C = +1$

Among the trajectories for which $C = +1$ are those which start, with velocity different from zero, at the libration point $x = 0$, $y = 0$. Under these conditions, the Jacobi integral becomes

$$\dot{x}_o^2 + \dot{y}_o^2 = 1$$

These trajectories constitute a one-parameter family of solutions that can be parameterized by $\dot{x}_o = \cos \gamma$ and $\dot{y}_o = \sin \gamma$, where γ is the parameter. The values of K_i are

$$K_1 = \frac{2 \sin \gamma}{\left(\lambda_2^2 - \lambda_1^2\right)}$$

$$K_2 = -\frac{\lambda_1\left(\omega_1^2 + \lambda_2^2\right)\cos \gamma}{\omega_1^2\left(\lambda_2^2 - \lambda_1^2\right)}$$

$$K_3 = -\frac{2 \sin \gamma}{\left(\lambda_2^2 - \lambda_1^2\right)}$$

$$K_4 = \frac{\lambda_2\left(\omega_1^2 + \lambda_1^2\right)\cos \gamma}{\omega_1^2\left(\lambda_2^2 - \lambda_1^2\right)}$$

Typical trajectories are drawn for $\gamma = 0°$, $45°$, $90°$, $135°$, $180°$, $225°$, $270°$, and $315°$ in figure 2.

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[3]This corresponds to approximately 217.422 days.

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5.3 Trajectories for Which $C = 0$

The third possible value for C is zero. Under these conditions, the Jacobi integral becomes

$$\dot{x}_o{}^2 + \dot{y}_o{}^2 = \omega_1{}^2 x_o{}^2 + \omega_2{}^2 y_o{}^2$$

The trajectories to be used as models will be those that start on the ellipse

$$\omega_1{}^2 x_o{}^2 + \omega_2{}^2 y_o{}^2 = \alpha^2$$

with the proper initial velocity dictated by the equation

$$\dot{x}_o{}^2 + \dot{y}_o{}^2 = \alpha^2$$

These conditions may be parameterized by $x_o = (\alpha/\omega_1) \cos \beta$, $y_o = (\alpha/\omega_2) \sin \beta$, $\dot{x}_o = \alpha \cos \gamma$, and $\dot{y}_o = \alpha \sin \gamma$. The values of K_i are

$$K_1 = \frac{\alpha\left[\left(\omega_1{}^2 + \lambda_2{}^2\right) \cos \beta + 2\omega_1 \sin \gamma\right]}{\omega_1\left(\lambda_2{}^2 - \lambda_1{}^2\right)}$$

$$K_2 = \frac{\alpha\lambda_1}{\omega_1{}^2\omega_2}\left[\frac{2\lambda_2{}^2 \sin \beta - \left(\omega_1{}^2 + \lambda_2{}^2\right)\omega_2 \cos \gamma}{\lambda_2{}^2 - \lambda_1{}^2}\right]$$

$$K_3 = -\frac{\alpha\left[\left(\omega_1{}^2 + \lambda_1{}^2\right) \cos \beta + 2\omega_1 \sin \gamma\right]}{\omega_1\left(\lambda_2{}^2 - \lambda_1{}^2\right)}$$

$$K_4 = \frac{\alpha\lambda_2}{\omega_1{}^2\omega_2}\left[\frac{\left(\omega_1{}^2 + \lambda_1{}^2\right)\omega_2 \cos \gamma - 2\lambda_1{}^2 \sin \beta}{\lambda_2{}^2 - \lambda_1{}^2}\right]$$

Typical trajectories are shown in figure 3 for $\beta, \gamma = 0°, 45°, 90°, 135°, 180°, 225°, 270°,$ and $315°$. With $\alpha = 1$, most of the trajectories lay within the bounds of the size of the paper available (figs. 3(a) to 3(h)). For those trajectories that exceeded these bounds, the drawings were rescaled by choosing $\alpha = 0.5$; these trajectories are given in green (figs. 3(i) to 3(n)).

6.0 ENVELOPES

For the trajectories drawn, there is an exterior bounding curve for each. In section 6.1, a geometric description is given that enables determination of this bounding curve. The equation for the bounding curve, found as an envelope, is given in section 6.2. In section 6.3, orbits of spaceship observatories are described. An approximation to the bounding curve had been known; in section 6.4, this approximation is compared to the envelope determined in section 6.2. From the analysis given in section 6.2, an inner region from which the trajectory may be excluded can be determined; this is done in section 6.5. It was observed that, for all the trajectories drawn, $J_2 \geqslant J_1$. The question thus arises: Is this always true or are there initial conditions for which $J_1 > J_2$? This is answered affirmatively in section 6.6 and trajectories for a select set of initial conditions are drawn.

6.1 A Geometric Description

Consider a ray emanating from the libration point and the intersections of the trajectory with it. If the ray is referenced by an angle θ, let $r(\theta)$ be the least upper bound of all the intersections. Then, $r = r(\theta)$ is the bounding curve for the trajectory.

On a fixed ray r, the distances from the libration point $x = y = 0$ to successive intersections with the trajectory seem randomly generated. To proceed directly from this sequence of distances to their least upper bound seems hopeless; an alternate description is needed.

Consider a particle P moving according to equations (4a) and (4b) where the K_i are given by equations (5a) to (5d) for the initial conditions (x_o, y_o), $(\dot{x}_o, \dot{y}_o)$. At $t = 0$, when P is launched, a spaceship S_1 is also launched. In describing the motion of S_1, whose equations of motion will also be equations (4a) and (4b), K_1^*, K_2^*, K_3^*, and K_4^* shall be used to denote the coefficients. The initial conditions for S_1 will be represented by (x_o^*, y_o^*), $(\dot{x}_o^*, \dot{y}_o^*)$. The space-

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ship S_1 is launched so that

$$\dot{y}_o^* = -\frac{\left(\omega_1^2 + \lambda_1^2\right)x_o^*}{2}$$

and

$$y_o^* = \frac{\left(\omega_1^2 + \lambda_1^2\right)\dot{x}_o^*}{2\lambda_1^2}$$

i.e., the initial conditions for spaceship S_1 are chosen so as to make $K_3^* = K_4^* = 0$.

Under these conditions, the trajectory of the spaceship S_1 is given by

$$x^* = K_1^* \cos \lambda_1 t + K_2^* \sin \lambda_1 t$$

$$y^* = M_1 \left(K_2^* \cos \lambda_1 t - K_1^* \sin \lambda_1 t\right)$$

where $K_1^* = x_o^*$ and $K_2^* = \dot{x}_o^*/\lambda_1$. If x_o^* and $\dot{x}_o^*$ are chosen as $x_o^* = K_1$ and $\dot{x}_o^* = \lambda_1 K_2$, then $K_1^* \equiv K_1$ and $K_2^* \equiv K_2$. Thus, for appropriately chosen initial conditions, the trajectory of spaceship S_1 is the ellipse E_1

$$x^* = K_1 \cos \lambda_1 t + K_2 \sin \lambda_1 t$$

$$= J_1 \cos \left(\lambda_1 t - \phi_1\right)$$

$$y^* = M_1 \left(K_2 \cos \lambda_1 t - K_1 \sin \lambda_1 t\right)$$

$$= M_1 J_1 \sin \left(\lambda_1 t - \phi_1\right)$$

As viewed by an observer on S_1, the particle P describes the ellipse E_3

$$\bar{x} = K_3 \cos \lambda_2 t + K_4 \sin \lambda_2 t$$

$$= J_2 \cos \left(\lambda_2 t - \phi_2\right)$$

$$\bar{y} = M_2\left(K_4 \cos \lambda_2 t - K_3 \sin \lambda_2 t\right)$$

$$= M_2 J_2 \sin \left(\lambda_2 t - \phi_2\right)$$

The semi-major and semi-minor axes for E_1 are J_1 and $M_1 J_1$, respectively; for E_3, they are J_2 and $M_2 J_2$, respectively. An observer fixed in the (x,y) system and looking at S_1 and P can visualize the ellipse E_3 being attached to S_1 and moving with it. The center of E_3 is at S_1 and the axes are parallel to the axes of E_1, as shown in the following sketch.

The ellipse E_3 rides on the ellipse E_1
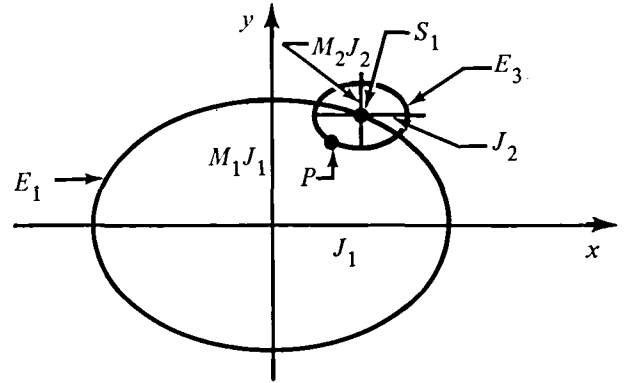
Again, consider the particle P moving according to equations (4a) and (4b) where the K_i are given by equations (5a) to (5d) for initial conditions (x_o, y_o), $(\dot{x}_o, \dot{y}_o)$. At $t = 0$, when P is launched, a second spaceship S_2 is also launched. In describing the motion of S_2, whose equations of motion shall also be equations (4a) and (4b), K_1^{**}, K_2^{**}, K_3^{**}, and K_4^{**} shall be used to denote the coefficients. The initial conditions for S_2 will be represented by (x_o^{**}, y_o^{**}), $(\dot{x}_o^{**}, \dot{y}_o^{**})$. The

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spaceship S_2 is launched so that

$$\dot{y}_o^{**} = -\frac{\left(\omega_1^2 + \lambda_2^2\right) x_o^{**}}{2}$$

and

$$y_o^{**} = \frac{\left(\omega_1^2 + \lambda_2^2\right) \dot{x}_o^{**}}{2\lambda_2^2}$$

i.e., the initial conditions for spaceship S_2 are chosen so as to make $K_1^{**} = K_2^{**} = 0$. The trajectory for spaceship S_2 is given by the equations

$$x^{**} = K_3^{**} \cos \lambda_2 t + K_4^{**} \sin \lambda_2 t$$

$$y^{**} = M_2\left(K_4^{**} \cos \lambda_2 t - K_3^{**} \sin \lambda_2 t\right)$$

where $K_3^{**} = x_o^{**}$ and $K_4^{**} = \dot{x}_o^{**}/\lambda_2$. If x_o^{**} and $\dot{x}_o^{**}$ are chosen as $x_o^{**} = K_3$ and $\dot{x}_o^{**} = \lambda_2 K_4$, then $K_3^{**} \equiv K_3$ and $K_4^{**} \equiv K_4$. Thus, for appropriately chosen initial conditions, the trajectory of spaceship S_2 is the ellipse E_2

$$x^{**} = K_3 \cos \lambda_2 t + K_4 \sin \lambda_2 t$$

$$= J_2 \cos \left(\lambda_2 t - \phi_2\right)$$

$$y^{**} = M_2\left(K_4 \cos \lambda_2 t - K_3 \sin \lambda_2 t\right)$$

$$= M_2 J_2 \sin \left(\lambda_2 t - \phi_2\right)$$

and, when viewed by an observer on S_2, the particle

P describes the ellipse E_4

$$\bar{\bar{x}} = K_1 \cos \lambda_1 t + K_2 \sin \lambda_1 t$$

$$= J_1 \cos \left(\lambda_1 t - \phi_1\right)$$

$$\bar{\bar{y}} = M_1\left(K_2 \cos \lambda_1 t - K_1 \sin \lambda_1 t\right)$$

$$= M_1 J_1 \sin \left(\lambda_1 t - \phi_1\right)$$

The semi-major and semi-minor axes for E_2 are J_2 and $M_2 J_2$, respectively; for E_4, they are J_1 and $M_1 J_1$, respectively. The geometry for this situation is illustrated in the following sketch.

The ellipse E_4 rides on the ellipse E_2

In the subsequent discussions, the description will be based on the image of the ellipse with the smaller semi-major axis riding on that spaceship whose trajectory has the larger semi-major axis. In effect, J_1 and J_2 are being compared, since $M_1 \approx 0.49 \ldots$ and $M_2 \approx 0.19 \ldots$ for $\mu = 1/82.45$.

6.2 The Equation of the Envelope

When a fixed ray emanating from the origin intersects the smaller ellipse as it is being carried piggyback on the larger ellipse, the farthest point of intersection is an upper bound for all intersections of that ray and the trajectory. This farthest point of intersection can be found quantitatively.

Assume that $J_1 > J_2$. The equation of the ellipse which the spaceship S_1 traverses is $x = J_1 \cos \alpha$ and $y = M_1 J_1 \sin \alpha$, where α, the eccentric angle, is considered a parameter.[4] The equation of the smaller ellipse on whose perimeter the particle rides is

$$\frac{\left(x - J_1 \cos \alpha\right)^2}{J_2{}^2} + \frac{\left(y - M_1 J_1 \sin \alpha\right)^2}{M_2{}^2 J_2{}^2} = 1 \quad (17)$$

If the envelope of this one-parameter family of ellipses is determined, a bound for the trajectory will be found. To find the envelope, first differentiate equation (17) with respect to α. This produces

$$\frac{J_1}{J_2{}^2}(x - J_1 \cos \alpha) \sin \alpha - \frac{M_1 J_1}{M_2{}^2 J_2{}^2}(y - M_1 J_1 \sin \alpha) \cos \alpha = 0$$

which can be simplified to

$$\left(x - J_1 \cos \alpha\right) \sin \alpha - \frac{M_1}{M_2{}^2}(y - M_1 J_1 \sin \alpha) \cos \alpha = 0$$

$$(18)$$

It remains to solve equations (17) and (18) for x and y. Solve equation (18) for $(x - J_1 \cos \alpha)$ and substitute this into equation (17). The resulting equation can be solved for y; there are two solutions:

$$y^{\#} = M_1 J_1 \sin \alpha \pm \frac{M_2{}^2 J_2 \sin \alpha}{\sqrt{M_1{}^2 \cos^2\alpha + M_2{}^2 \sin^2\alpha}} \quad (19a)$$

When these two values of y are substituted into equation (18), two values of x are determined.

$$x^{\#} = J_1 \cos \alpha \pm \frac{M_1 J_2 \cos \alpha}{\sqrt{M_1{}^2\cos^2\alpha + M_2{}^2\sin^2\alpha}} \quad (19b)$$

It is important to remember that the plus signs go together as do the minus signs. Equations (19a) and (19b) are the parametric equations for the envelope. Using the plus signs, the bounding curve is drawn in blue in figures 1 to 3.

The analysis for $J_2 \geqslant J_1$ parallels that given above and leads to the equations[5]

$$x^{\#} = J_2 \cos \alpha \pm \frac{M_2 J_1 \cos \alpha}{\sqrt{M_2{}^2\cos^2\alpha + M_1{}^2\sin^2\alpha}}$$

$$y^{\#} = M_2 J_2 \sin \alpha \pm \frac{M_1{}^2 J_1 \sin \alpha}{\sqrt{M_2{}^2\cos^2\alpha + M_1{}^2\sin^2\alpha}}$$

6.3 The Orbit of the Spaceship

Each spaceship, because of the special way the initial conditions were chosen (i.e., for S_1, $J_2 = 0$; for S_2, $J_1 = 0$), has a periodic orbit, an ellipse. These elliptic orbits are described in reference 2 (pp. 258-261).

6.4 An Approximation to the Envelope

An approximation to the envelope developed in section 6.2 can be obtained by arbitrarily assuming the envelope to be an ellipse with semi-major axis $J_1 + J_2$ and semi-minor axis $M_1 J_1 + M_2 J_2$ (ref. 2, p. 283). A comparison of this approximate bound and the bound found in section 6.2 is shown in figure 4. The bounding curve is drawn in blue and the ellipse in red. At the scale in which the figures are drawn, the approximation is remarkable.

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[4]Note that $\alpha = \lambda_1 t - \phi_1$.

[5]Note that, in this case, $\alpha = \lambda_2 t - \phi_2$.

6.5 An Inner Bounding Curve

In section 6.2, an outer bounding curve was obtained when the plus sign was used in equations (19a) and (19b). From the geometry of a family of ellipses whose centers lie on a fixed ellipse, one suspects that, under the right conditions, there may be an inner region from which the trajectory is excluded. The existence of such an inner region depends on the relative magnitude of J_1 and J_2 and the relative magnitude of $M_1 J_1$ and $M_2 J_2$.

The case of $C = 0$ is disposed of easily. From equation (16), it is found that

$$J_1^2 = \left(\frac{\lambda_2^2 - \omega_2^2 M_2^2}{\omega_2^2 M_1^2 - \lambda_1^2} \right) J_2^2$$
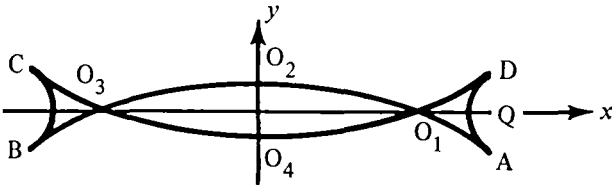
and for $\mu = 1/82.45$,

$$J_1 = 0.3515131989732791 \, J_2$$

and

$$(M_1 J_1) = 0.8878271396217024 (M_2 J_2)$$

In this case (refer to the sketch on page 12), there is always an inner region from which the trajectory is excluded. Equations (19a) and (19b), when the minus sign is used, produce a curve that looks like the one shown in the following sketch.

For the parameter α (α going from 0 to 2π), the sequence to be followed in tracing the curve is $QAO_1 O_2 O_3 BCO_3 O_4 O_1 DQ$. However, the two triangular regions, $AO_1 D$ and $BO_3 C$, are *not* part of the excluded region; the excluded region is bounded by $O_1 O_2 O_3 O_4$. A typical situation is shown in figure 5. In figure 6, the family of smaller ellipses is drawn in black, the trajectory in green, and the bounding curves in red.

For $C = +1$, there may be no inner excluded region, as can be seen in figure 7. The inner envelope in this illustration, drawn in green, does not encompass a region from which the trajectory is excluded. In the next section, initial conditions are given for which there does exist an inner region from which the trajectory is excluded.

For $C = -1$, there will always be an inner excluded region that includes the Jacobi ellipse.

6.6 On the Relative Magnitude of J_1 and J_2

Equation (16) can be rewritten as

$$J_1^2 = \frac{(\omega_2^2 M_2^2 - \lambda_2^2)}{(\lambda_1^2 - \omega_2^2 M_1^2)} J_2^2 + \frac{C}{(\lambda_1^2 - \omega_2^2 M_1^2)}$$

For $\mu = 1/82.45$,

$$J_1^2 = (0.1235615290524281) J_2^2 + (5.182189180103332) C$$

For all the figures drawn thus far, $J_1 \leqq J_2$; are there initial conditions for which $J_1 > J_2$? If $C = 0$,

$$J_1 = (0.3515131989732791) J_2$$

and clearly $J_1 < J_2$. Similarly, if $C = -1$,

$$J_1 < (0.3515131989732791) J_2$$

Thus, if any trajectories are to be found for which $J_1 > J_2$, they can only be among those for which $C = +1$.

It is not intended to exhibit samples of all trajectories for which $J_1 > J_2$, only some of a special set. If J_1

$> J_2$, then $J_1{}^2 > J_2{}^2$ and $K_1{}^2 + K_2{}^2 > K_3{}^2 + K_4{}^2$. Using the parametric representation of equation (22) in the appendix,

$$K_1 = \frac{\left(\omega_1{}^2 + \lambda_2{}^2\right)\cosh\alpha\sinh\beta + 2\omega_1\cosh\alpha\cosh\beta\sin\gamma}{\omega_1\left(\lambda_2{}^2 - \lambda_1{}^2\right)}$$

$$K_2 = \frac{\lambda_1}{\omega_1{}^2}\left[\frac{2\lambda_2{}^2\sinh\alpha - \omega_2\left(\omega_1{}^2 + \lambda_2{}^2\right)\cosh\alpha\cosh\beta\cos\gamma}{\omega_2\left(\lambda_2{}^2 - \lambda_1{}^2\right)}\right]$$

$$K_3 = -\frac{\left(\omega_1{}^2 + \lambda_1{}^2\right)\cosh\alpha\sinh\beta + 2\omega_1\cosh\alpha\cosh\beta\sin\gamma}{\omega_1\left(\lambda_2{}^2 - \lambda_1{}^2\right)}$$

$$K_4 = \frac{\lambda_2}{\omega_1{}^2}\left[\frac{\omega_2\left(\omega_1{}^2 + \lambda_1{}^2\right)\cosh\alpha\cosh\beta\cos\gamma - 2\lambda_1{}^2\sinh\alpha}{\omega_2\left(\lambda_2{}^2 - \lambda_1{}^2\right)}\right]$$

Special values are chosen for α and γ; $\alpha = 0$, $\gamma = 90°$. For these values,

$$K_1 = \frac{\left(\omega_1{}^2 + \lambda_2{}^2\right)\sinh\beta + 2\omega_1\cosh\beta}{\omega_1\left(\lambda_2{}^2 - \lambda_1{}^2\right)}$$

$$K_2 = 0$$

$$K_3 = -\frac{\left(\omega_1{}^2 + \lambda_1{}^2\right)\sinh\beta + 2\omega_1\cosh\beta}{\omega_1\left(\lambda_2{}^2 - \lambda_1{}^2\right)}$$

$$K_4 = 0$$

Thus, $J_1 > J_2$ implies that $|K_1| > |K_3|$.

Case 1

$$\left[\left(\omega_1{}^2 + \lambda_2{}^2\right)\sinh\beta + 2\omega_1\cosh\beta\right] >$$

$$-\left[\left(\omega_1{}^2 + \lambda_1{}^2\right)\sinh\beta + 2\omega_1\cosh\beta\right]$$

This implies that

$$\tanh\beta > -\frac{4\omega_1}{2\omega_1{}^2 + \lambda_1{}^2 + \lambda_2{}^2}$$

or that $\beta > -0.7343384813667741$.

Case 2

$$\left[\left(\omega_1{}^2 + \lambda_2{}^2\right)\sinh\beta + 2\omega_1\cosh\beta\right] >$$

$$\left[\left(\omega_1{}^2 + \lambda_1{}^2\right)\sinh\beta + 2\omega_1\cosh\beta\right]$$

This implies that

$$\left(\lambda_2{}^2 - \lambda_1{}^2\right)\sinh\beta > 0$$

or, since $(\lambda_2{}^2 - \lambda_1{}^2) < 0$, that $\sinh\beta < 0$. However, $\sinh\beta < 0$ for $\beta < 0$. Therefore, we find that $J_1 > J_2$ for $\alpha = 0$, $-0.73 < \beta < 0$, $\gamma = 90°$. Trajectories for $\beta = -0.1$ through -0.7 are given in figure 8. The shape of the interior excluded region in figure 8, as shown in the following sketch, is similar to that of the sketch shown in section 6.5 rotated 90°.

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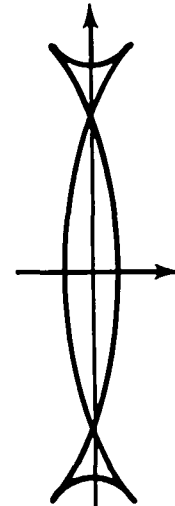
7.0 CONCLUDING REMARKS

The derivation of the finite equations of motion from the differential equations that describe the motion of a point particle in the neighborhood of a triangular libration point (L_4 or L_5) in the linearized, restricted problem of three bodies is well known. From these equations, three integral invariants of the motion were derived; the Jacobi integral is expressed linearly in terms of two of these.

The trajectories for varied initial conditions were drawn. The motion of a point particle as it traversed its trajectory was described in elementary geometric terms.

Each trajectory has an exterior boundary curve; its equation was found. An approximation to this boundary curve had been known; it is a well-defined ellipse. For different initial conditions, the boundary curve and the ellipse were compared graphically; the agreement seems remarkable. For certain initial conditions, there is an interior region from which the trajectory is excluded; the equation of the boundary of this region was found.

Lyndon B. Johnson Space Center
National Aeronautics and Space Administration
Houston, Texas, September 28, 1979
953-36-00-00-72

8.0 REFERENCES

1. Moulton, F. R.: An Introduction to Celestial Mechanics. The MacMillan Co. (New York), 1939, pp. 277-281 and 298-307.
2. Szebehely, Victor: Theory of Orbits. Academic Press (New York), 1967.

Appendix
Parameterizations for the Jacobi Ellipse

A convenient method for choosing the initial values of the position and velocity of the particle can be obtained from the Jacobi integral:

$$\dot{x}_o{}^2 + \dot{y}_o{}^2 - \omega_1{}^2 x_o{}^2 - \omega_2{}^2 y_o{}^2 = C \qquad (20)$$

If $C = -1$, equation (20) can be parameterized by

$$x_o = \frac{1}{\omega_1} \cosh \alpha \cosh \beta \cos \gamma \qquad (21a)$$

$$y_o = \frac{1}{\omega_2} \cosh \alpha \cosh \beta \sin \gamma \qquad (21b)$$

$$\dot{x}_o = \cosh \alpha \sinh \beta \qquad (21c)$$

$$\dot{y}_o = \sinh \alpha \qquad (21d)$$

For the rest trajectories (sec. 5.1), $\alpha = \beta = 0$, γ arbitrary. If the initial velocity is not zero, α and β can be determined from equations (21c) and (21d) and γ from equations (21a) and (21b).

If $C = +1$, equation (20) can be parameterized by

$$x_o = \frac{1}{\omega_1} \cosh \alpha \sinh \beta \qquad (22a)$$

$$y_o = \frac{1}{\omega_2} \sinh \alpha \qquad (22b)$$

$$\dot{x}_o = \cosh \alpha \cosh \beta \cos \gamma \qquad (22c)$$

$$\dot{y}_o = \cosh \alpha \cosh \beta \sin \gamma \qquad (22d)$$

For trajectories that start at the libration point ($x_o = y_o = 0$), $\alpha = \beta = 0$, γ arbitrary. If the initial position is different from the origin, α and β can be determined from equations (22a) and (22b) and γ from equations (22c) and (22d).

If $C = 0$, equation (20) can be parameterized by

$$x_o = \frac{1}{\omega_1} \alpha \cos \beta \qquad (23a)$$

$$y_o = \frac{1}{\omega_2} \alpha \sin \beta \qquad (23b)$$

$$\dot{x}_o = \alpha \cos \gamma \qquad (23c)$$

$$\dot{y}_o = \alpha \cos \gamma \qquad (23d)$$

Included among these trajectories is the solution corresponding to $\alpha = 0$; i.e., the particle is at the libration point with zero initial velocity and remains there indefinitely. Leaving this solution aside, the parameterization is as given above.

$$x_o = 6.062834463 \qquad y_o = 0$$
$$\dot{x}_o = 0 \qquad \dot{y}_o = 0$$
$$K_1 = -0.8547469351 \qquad K_2 = 0$$
$$K_3 = 6.917581398 \qquad K_4 = 0$$
$$J_1 = 0.8547469351 \qquad J_2 = 6.917581398$$
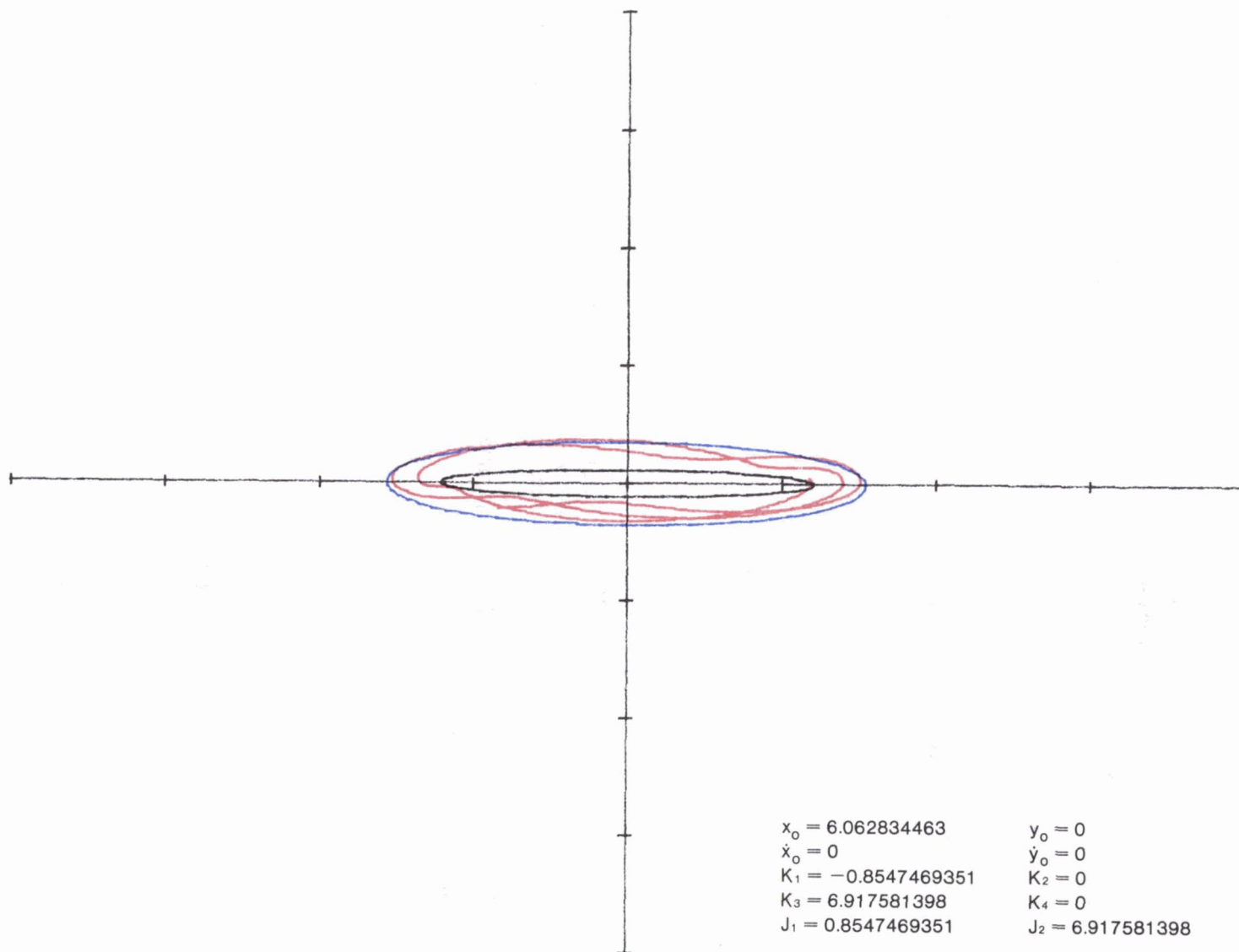
(a) $\alpha = 0$, $\beta = 0$, $\gamma = 0°$.

FIGURE 1.—Typical trajectories for which $C = -1$.

$x_o = 5.856248389$ $y_o = 0.1501114254$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = -0.8256221396$ $K_2 = -1.136727243$
$K_3 = 6.681870528$ $K_4 = 3.642388070$
$J_1 = 1.404920191$ $J_2 = 7.610150104$

(b) $\alpha = 0$, $\beta = 0$, $\gamma = 15°$.

FIGURE 1.—Continued.

$x_o = 5.250568664$ $y_o = 0.2899930053$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = -0.7402325596$ $K_2 = -2.195988403$
$K_3 = 5.990801224$ $K_4 = 7.036553412$
$J_1 = 2.317392783$ $J_2 = 9.241362628$

(c) $\alpha = 0$, $\beta = 0$, $\gamma = 30°$.

FIGURE 1.—Continued.

$x_o = 4.287071362$ $y_o = 0.4101120411$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = -0.6043973540$ $K_2 = -3.105596582$
$K_3 = 4.891468716$ $K_4 = 9.951189268$
$J_1 = 3.163862559$ $J_2 = 11.08840088$

(d) $\alpha = 0$, $\beta = 0$, $\gamma = 45°$.

FIGURE 1.—Continued.

$$x_o = 3.031417232 \qquad y_o = 0.5022826190$$
$$\dot{x}_o = 0 \qquad \dot{y}_o = 0$$
$$K_1 = -0.4273734676 \qquad K_2 = -3.803563487$$
$$K_3 = 3.458790699 \qquad K_4 = 12.18766802$$
$$J_1 = 3.827498306 \qquad J_2 = 12.66895753$$

(e) $\alpha = 0$, $\beta = 0$, $\gamma = 60°$.

FIGURE 1.—Continued.

$x_o = 1.569177026$ $y_o = 0.5602234665$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = -0.2212247855$ $K_2 = -4.242323825$
$K_3 = 1.790401812$ $K_4 = 13.59357734$
$J_1 = 4.248088022$ $J_2 = 13.71097675$

(f) $\alpha = 0$, $\beta = 0$, $\gamma = 75°$.

FIGURE 1.—Continued.

$x_0 = 0$ $y_0 = 0.5799860106$
$\dot{x}_0 = 0$ $\dot{y}_0 = 0$
$K_1 = 0$ $K_2 = -4.391976806$
$K_3 = 0$ $K_4 = 14.07310682$
$J_1 = 4.391976806$ $J_2 = 14.07310682$

(g) $\alpha = 0$, $\beta = 0$, $\gamma = 90°$.

FIGURE 1.—Continued.

$x_o = -1.569177026$ $y_o = 0.5602234665$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = 0.2212247855$ $K_2 = -4.242323825$
$K_3 = -1.790401812$ $K_4 = 13.59357734$
$J_1 = 4.248088022$ $J_2 = 13.71097675$

(h) $\alpha = 0$, $\beta = 0$, $\gamma = 105°$.

FIGURE 1.—Continued.

$x_o = -3.031417232$ $y_o = 0.5022826190$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = 0.4273734676$ $K_2 = -3.803563487$
$K_3 = -3.458790699$ $K_4 = 12.18766802$
$J_1 = 3.827498306$ $J_2 = 12.66895753$

(i) $\alpha = 0$, $\beta = 0$, $\gamma = 120°$.

FIGURE 1.—Continued.

$x_o = -4.287071362$ $y_o = 0.4101120411$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = 0.6043973540$ $K_2 = -3.105596582$
$K_3 = -4.891468716$ $K_4 = 9.951189268$
$J_1 = 3.163862559$ $J_2 = 11.08840088$

(j) $\alpha = 0$, $\beta = 0$, $\gamma = 135°$.

FIGURE 1.—Continued.

$x_o = -5.250568664$ $y_o = 0.2899930053$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = 0.7402325596$ $K_2 = -2.195988403$
$K_3 = -5.990801224$ $K_4 = 7.036553412$
$J_1 = 2.317392783$ $J_2 = 9.241362628$

(k) $\alpha = 0$, $\beta = 0$, $\gamma = 150°$.

FIGURE 1.—Continued.

$x_o = -5.856248389$ $y_o = 0.1501114254$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = 0.8256221396$ $K_2 = -1.136727243$
$K_3 = -6.681870528$ $K_4 = 3.642388070$
$J_1 = 1.404920191$ $J_2 = 7.610150104$

(l) $\alpha = 0$, $\beta = 0$, $\gamma = 165°$.

FIGURE 1.—Continued.

$x_o = -6.062834463$ $y_o = 0$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = 0.8547469351$ $K_2 = 0$
$K_3 = -6.917581398$ $K_4 = 0$
$J_1 = 0.8547469351$ $J_2 = 6.917581398$

(m) $\alpha = 0$, $\beta = 0$, $\gamma = 180°$.

FIGURE 1.—Continued.

$x_o = -5.856248389$ $y_o = -0.1501114254$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = 0.8256221396$ $K_2 = 1.136727243$
$K_3 = -6.681870528$ $K_4 = -3.642388070$
$J_1 = 1.404920191$ $J_2 = 7.610150104$

(n) $\alpha = 0$, $\beta = 0$, $\gamma = 195°$.

FIGURE 1.—Continued.

$x_o = -5.250568664$ $y_o = -0.2899930053$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = 0.7402325596$ $K_2 = 2.195988403$
$K_3 = -5.990801224$ $K_4 = -7.036553412$
$J_1 = 2.317392783$ $J_2 = 9.241362628$

(o) $\alpha = 0$, $\beta = 0$, $\gamma = 210°$.

FIGURE 1.—Continued.

$x_0 = -4.287071362$ $y_0 = -0.4101120411$
$\dot{x}_0 = 0$ $\dot{y}_0 = 0$
$K_1 = 0.6043973540$ $K_2 = 3.105596582$
$K_3 = -4.891468716$ $K_4 = -9.951189268$
$J_1 = 3.163862559$ $J_2 = 11.08840088$

(p) $\alpha = 0$, $\beta = 0$, $\gamma = 225°$.

FIGURE 1.—Continued.

$x_o = -3.031417232$ $y_o = -0.5022826190$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = 0.4273734676$ $K_2 = 3.803563487$
$K_3 = -3.458790699$ $K_4 = -12.18766802$
$J_1 = 3.827498306$ $J_2 = 12.66895753$

(q) $\alpha = 0$, $\beta = 0$, $\gamma = 240°$.

FIGURE 1.—Continued.

$x_o = -1.569177026$ $y_o = -0.5602234665$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = 0.2212247855$ $K_2 = 4.242323825$
$K_3 = -1.790401812$ $K_4 = -13.59357734$
$J_1 = 4.248088022$ $J_2 = 13.71097675$

(r) $\alpha = 0$, $\beta = 0$, $\gamma = 255°$.

FIGURE 1.—Continued.

$x_o = 0$

$\dot{x}_o = 0$

$K_1 = 0$

$K_3 = 0$

$J_1 = 4.391976806$

$y_o = -0.5799860106$

$\dot{y}_o = 0$

$K_2 = 4.391976806$

$K_4 = -14.07310682$

$J_2 = 14.07310682$

(s) $\alpha = 0$, $\beta = 0$, $\gamma = 270°$.

FIGURE 1.—Continued.

$x_o = 1.569177026$ $y_o = -0.5602234665$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = -0.2212247855$ $K_2 = 4.242323825$
$K_3 = 1.790401812$ $K_4 = -13.59357734$
$J_1 = 4.248088022$ $J_2 = 13.71097675$

(t) $\alpha = 0$, $\beta = 0$, $\gamma = 285°$.

FIGURE 1.—Continued.

$x_o = 3.031417232$ $y_o = -0.5022826190$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = -0.4273734675$ $K_2 = 3.803563487$
$K_3 = 3.458790699$ $K_4 = -12.18766802$
$J_1 = 3.827498306$ $J_2 = 12.66895753$

(u) $\alpha = 0$, $\beta = 0$, $\gamma = 300°$.

FIGURE 1.—Continued.

$$x_o = 4.287071362 \qquad y_o = -0.4101120411$$
$$\dot{x}_o = 0 \qquad \dot{y}_o = 0$$
$$K_1 = -0.6043973540 \qquad K_2 = 3.105596582$$
$$K_3 = 4.891468716 \qquad K_4 = -9.951189268$$
$$J_1 = 3.163862559 \qquad J_2 = 11.08840088$$

(v) $\alpha = 0$, $\beta = 0$, $\gamma = 315°$.

FIGURE 1.—Continued.

$$x_o = 5.250568664 \qquad y_o = -0.2899930053$$
$$\dot{x}_o = 0 \qquad \dot{y}_o = 0$$
$$K_1 = -0.7402325596 \qquad K_2 = 2.195988403$$
$$K_3 = 5.990801224 \qquad K_4 = -7.036553412$$
$$J_1 = 2.317392783 \qquad J_2 = 9.241362628$$

(w) $\alpha = 0,\ \beta = 0,\ \gamma = 330°$.

FIGURE 1.—Continued.

$x_o = 5.856248389$ $y_o = -0.1501114254$
$\dot{x}_o = 0$ $\dot{y}_o = 0$
$K_1 = -0.8256221396$ $K_2 = 1.136727243$
$K_3 = 6.681870528$ $K_4 = -3.642388070$
$J_1 = 1.404920191$ $J_2 = 7.610150104$

(x) $\alpha = 0$, $\beta = 0$, $\gamma = 345°$.

FIGURE 1.—Concluded.

K₁ = 0   K₂ = 4.946882801
K₃ = 0   K₄ = −12.49448618
J₁ = 4.946882801   J₂ = 12.49448618
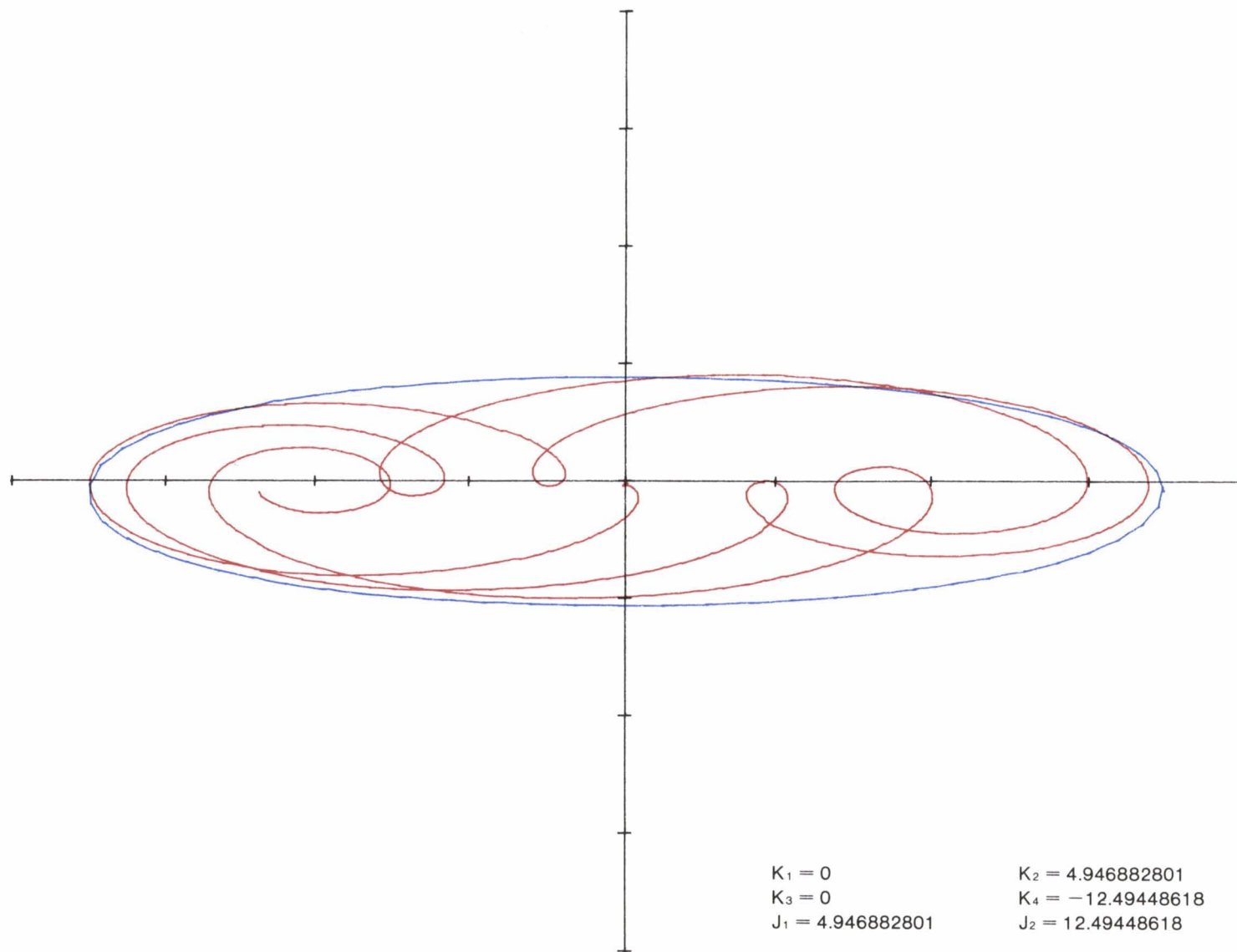
(a) $\alpha = 0$, $\beta = 0$, $\gamma = 0°$.

FIGURE 2.—Typical trajectories for which $C = +1$.

$K_1 = -1.719415817$ $K_2 = 3.497974374$
$K_3 = 1.719415817$ $K_4 = -8.834935904$
$J_1 = 3.897719266$ $J_2 = 9.000693483$

(b) $\alpha = 0$, $\beta = 0$, $\gamma = 45°$.

FIGURE 2.—Continued.

$K_1 = -2.431621168$ $K_2 = 0$
$K_3 = 2.431621168$ $K_4 = 0$
$J_1 = 2.431621168$ $J_2 = 2.431621168$

(c) $\alpha = 0$, $\beta = 0$, $\gamma = 90°$.

FIGURE 2.—Continued.

$K_1 = -1.719415817$ $K_2 = -3.497974374$
$K_3 = 1.719415817$ $K_4 = 8.834935904$
$J_1 = 3.897719266$ $J_2 = 9.000693483$

(d) $\alpha = 0$, $\beta = 0$, $\gamma = 135°$.

FIGURE 2.—Continued.

$K_1 = 0$ $K_2 = -4.946882801$
$K_3 = 0$ $K_4 = 12.49448618$
$J_1 = 4.946882801$ $J_2 = 12.49448618$

(e) $\alpha = 0$, $\beta = 0$, $\gamma = 180°$.

FIGURE 2.—Continued.

$K_1 = 1.719415817$ $K_2 = -3.497974374$
$K_3 = -1.719415817$ $K_4 = 8.834935904$
$J_1 = 3.897719266$ $J_2 = 9.000693483$

(f) $\alpha = 0, \beta = 0, \gamma = 225°$.

FIGURE 2.—Continued.

K₁ = 2.431621168   K₂ = 0
K₃ = −2.431621168   K₄ = 0
J₁ = 2.431621168   J₂ = 2.431621168

(g) $\alpha = 0$, $\beta = 0$, $\gamma = 270°$.

FIGURE 2.—Continued.

$K_1 = 1.719415817$ $K_2 = 3.497974374$
$K_3 = -1.719415817$ $K_4 = -8.834935904$
$J_1 = 3.897719266$ $J_2 = 9.000693483$

(h) $\alpha = 0$, $\beta = 0$, $\gamma = 315°$.

FIGURE 2.—Concluded.

K₁ = −0.8547469351   K₂ = 4.946882801
K₃ = 6.917581398   K₄ = −12.49448618
J₁ = 5.020183440   J₂ = 14.28163566

(a1) $\alpha = 1$, $\beta = 0°$, $\gamma = 0°$.
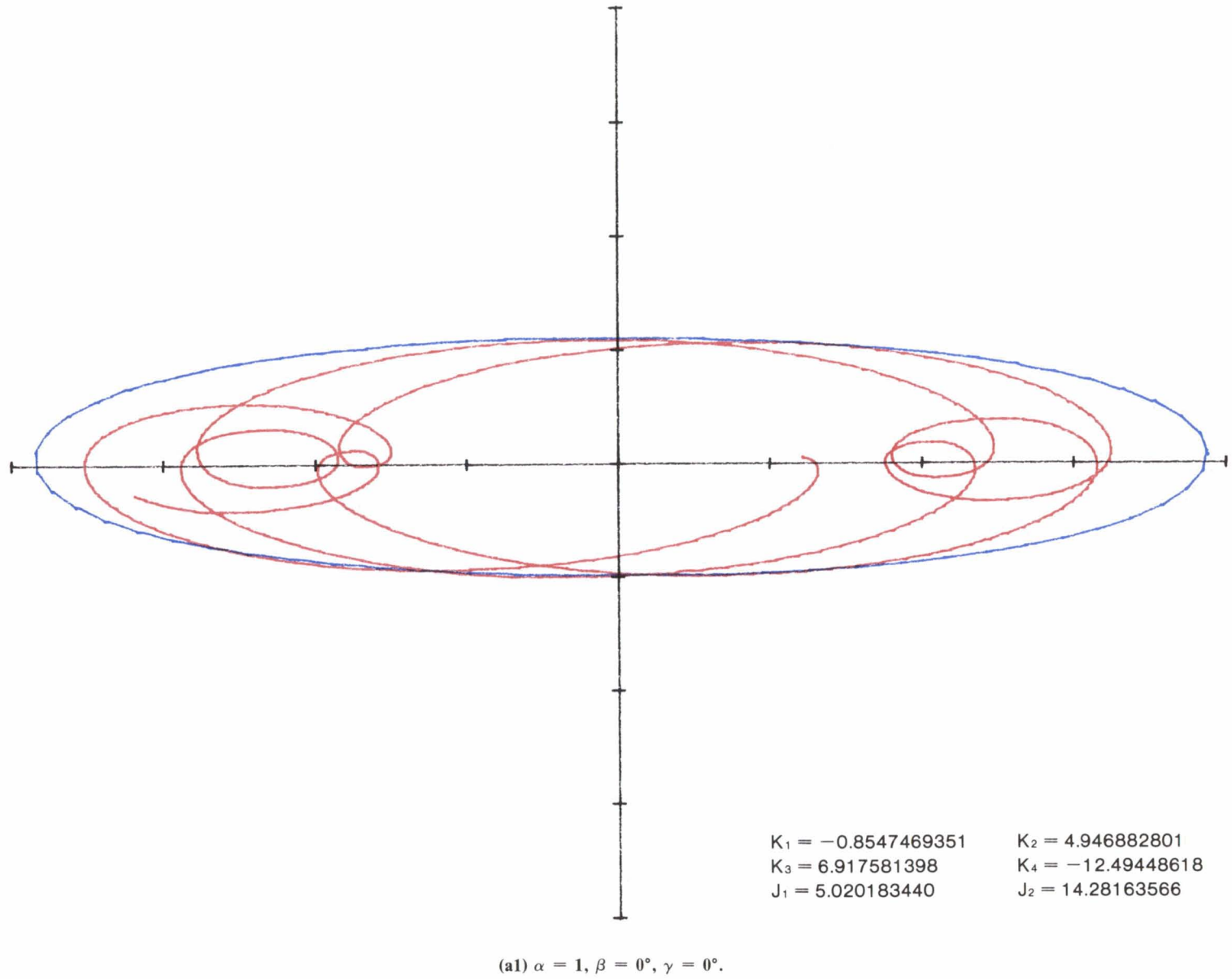
FIGURE 3.—Typical trajectories for which $C = 0$.

$K_1 = -0.6043973540$ $K_2 = 1.841286218$
$K_3 = 4.891468716$ $K_4 = -2.543296910$
$J_1 = 1.937945071$ $J_2 = 5.513150222$

(a2) $\alpha = 1$, $\beta = 0°$, $\gamma = 45°$.

FIGURE 3.—Continued.

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$K_1 = 0$ $K_2 = 0.5549059948$
$K_3 = 0$ $K_4 = 1.578620647$
$J_1 = 0.5549059948$ $J_2 = 1.578620647$

(a3) $\alpha = 1$, $\beta = 0°$, $\gamma = 90°$.

FIGURE 3.—Continued.

$K_1 = 0.6043973540$ $K_2 = 1.841286218$
$K_3 = -4.891468716$ $K_4 = -2.543296910$
$J_1 = 1.937945071$ $J_2 = 5.513150222$

(a4) $\alpha = 1$, $\beta = 0°$, $\gamma = 135°$.

FIGURE 3.—Continued.

$K_1 = 0.8547469351$ $K_2 = 4.946882801$
$K_3 = -6.917581398$ $K_4 = -12.49448618$
$J_1 = 5.020183440$ $J_2 = 14.28163566$

(a5) $\alpha = 1$, $\beta = 0°$, $\gamma = 180°$.

FIGURE 3.—Continued.

$K_1 = 0.6043973540$ $K_2 = 8.052479383$
$K_3 = -4.891468716$ $K_4 = -22.44567545$
$J_1 = 8.075129744$ $J_2 = 22.97247946$

(a6) $\alpha = 1$, $\beta = 0°$, $\gamma = 225°$.

FIGURE 3.—Continued.

$K_1 = 0$ $K_2 = 9.338859607$
$K_3 = 0$ $K_4 = -26.56759300$
$J_1 = 9.338859607$ $J_2 = 26.56759300$

(a7) $\alpha = 1$, $\beta = 0°$, $\gamma = 270°$.

FIGURE 3.—Continued.

$K_1 = -0.6043973540$ $K_2 = 8.052479383$
$K_3 = 4.891468716$ $K_4 = -22.44567545$
$J_1 = 8.075129744$ $J_2 = 22.97247946$

(a8) $\alpha = 1$, $\beta = 0°$, $\gamma = 315°$.

FIGURE 3.—Continued.

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K₁ = −2.574162752   K₂ = 3.497974374
K₃ = 8.636997215   K₄ = −8.834935904
J₁ = 4.343056366   J₂ = 12.35531518

(b1) $\alpha = 1$, $\beta = 45°$, $\gamma = 0°$.

FIGURE 3.—Continued.

$$K_1 = -2.323813171 \qquad K_2 = 0.3923777918$$
$$K_3 = 6.610884533 \qquad K_4 = 1.116253364$$
$$J_1 = 2.356707022 \qquad J_2 = 6.704462386$$

(b2) $\alpha = 1$, $\beta = 45°$, $\gamma = 45°$.

FIGURE 3.—Continued.

K₁ = −1.719415817   K₂ = −0.8940024318
K₃ = 1.719415817   K₄ = 5.238170921
J₁ = 1.937945071   J₂ = 5.513150220

(b3) $\alpha = 1$, $\beta = 45°$, $\gamma = 90°$.

FIGURE 3.—Continued.

$K_1 = -1.115018463$ $K_2 = 0.3923777918$
$K_3 = -3.172052899$ $K_4 = 1.116253364$
$J_1 = 1.182043360$ $J_2 = 3.362728233$

(b4) $\alpha = 1$, $\beta = 45°$, $\gamma = 135°$.

FIGURE 3.—Continued.

$K_1 = -0.8646688819$ $K_2 = 3.497974374$
$K_3 = -5.198165581$ $K_4 = -8.834935904$
$J_1 = 3.603259219$ $J_2 = 10.25070816$

(b5) $\alpha = 1$, $\beta = 45°$, $\gamma = 180°$.

FIGURE 3.—Continued.

$K_1 = -1.115018463$ $K_2 = 6.603570957$
$K_3 = -3.172052899$ $K_4 = -18.78612517$
$J_1 = 6.697045285$ $J_2 = 19.05204500$

(b6) $\alpha = 1$, $\beta = 45°$, $\gamma = 225°$.

FIGURE 3.—Continued.

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$K_1 = -1.719415817$ $K_2 = 7.889951180$
$K_3 = 1.719415817$ $K_4 = -22.90804273$
$J_1 = 8.075129744$ $J_2 = 22.97247946$

(b7) $\alpha = 1$, $\beta = 45°$, $\gamma = 270°$.

FIGURE 3.—Continued.

$K_1 = -2.323813171$ $K_2 = 6.603570957$
$K_3 = 6.610884533$ $K_4 = -18.78612517$
$J_1 = 7.000518340$ $J_2 = 19.91537831$

(b8) $\alpha = 1$, $\beta = 45°$, $\gamma = 315°$.

FIGURE 3.—Continued.

$K_1 = -3.286368103$ $K_2 = 0$
$K_3 = 9.349202566$ $K_4 = 0$
$J_1 = 3.286368103$ $J_2 = 9.349202566$
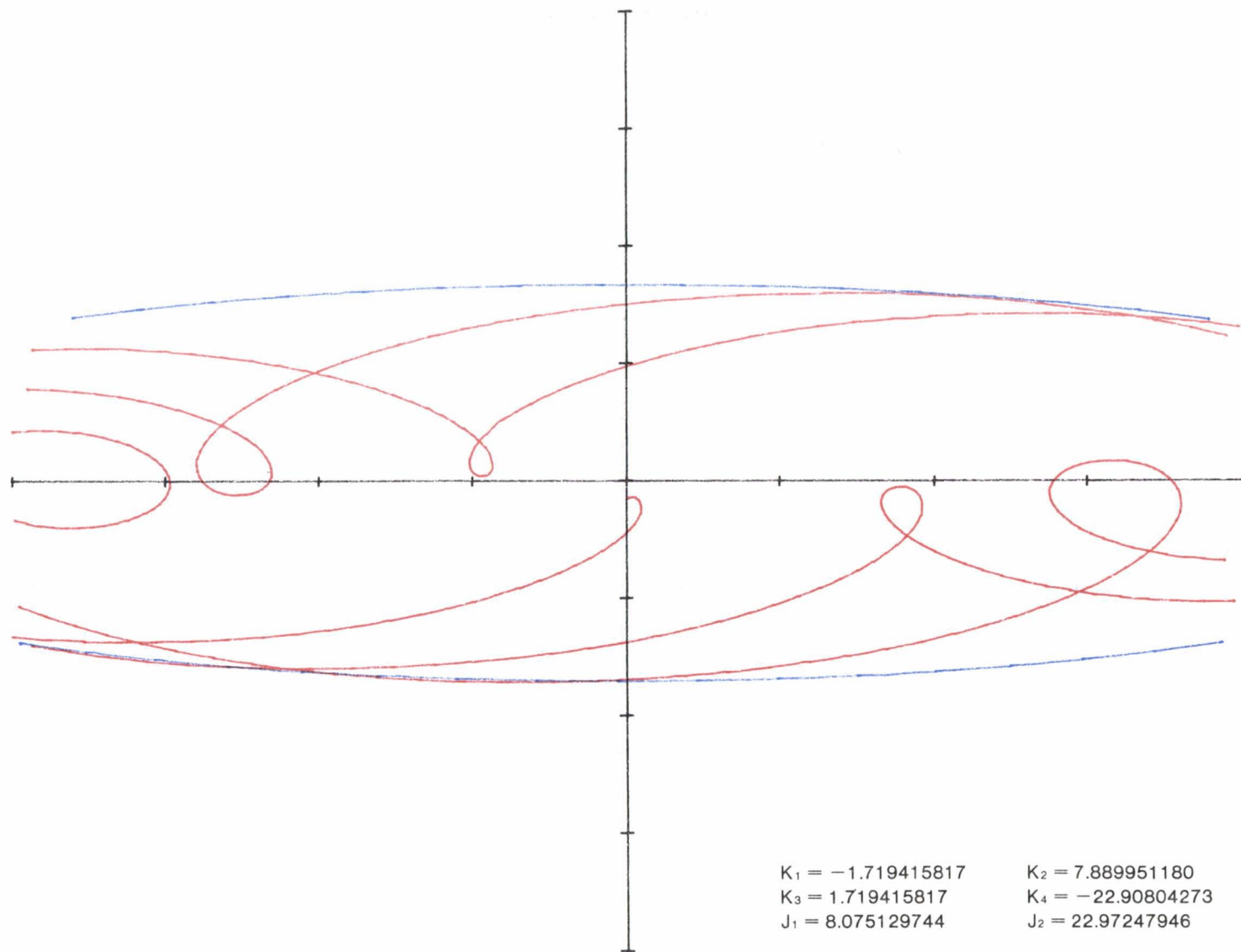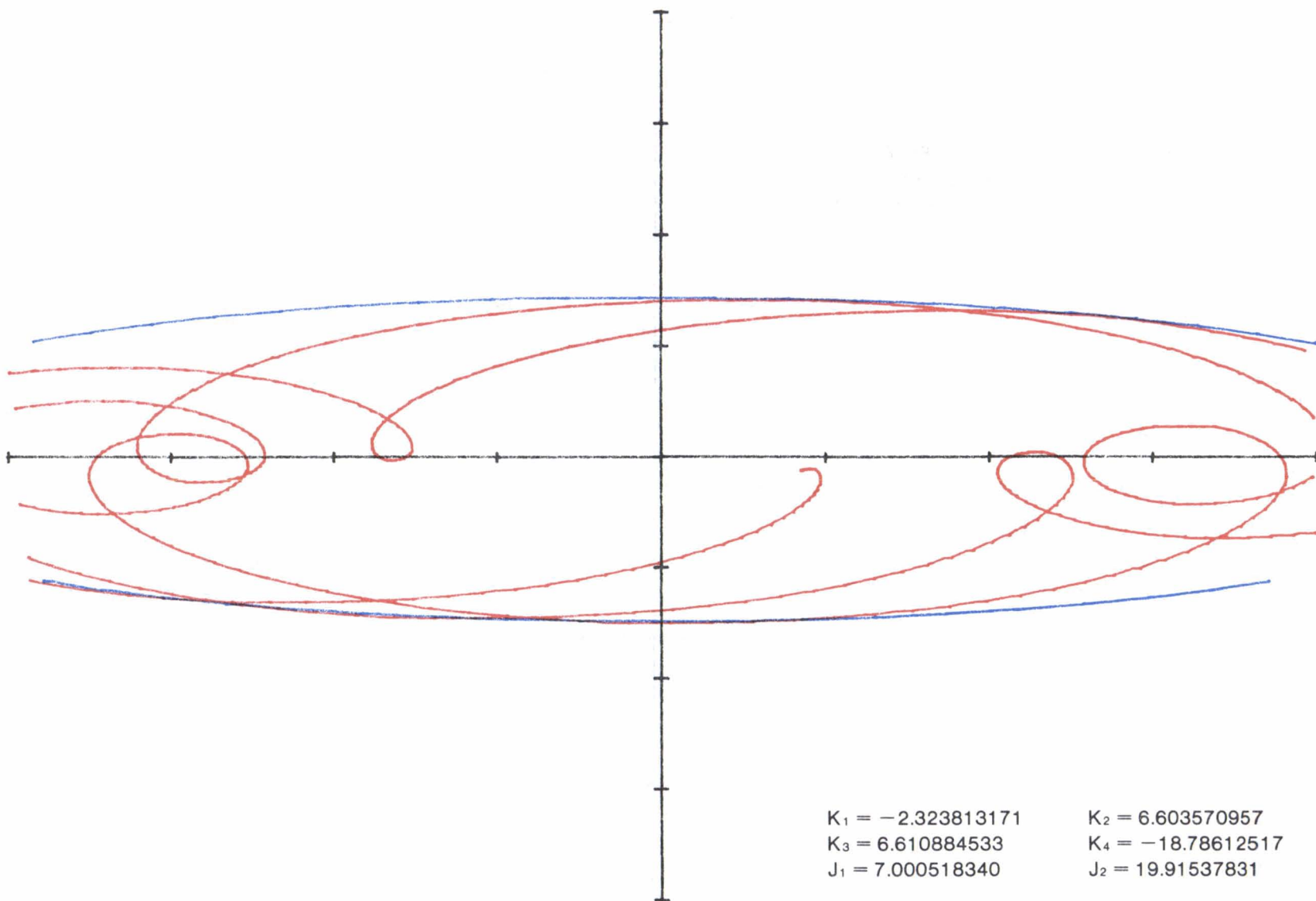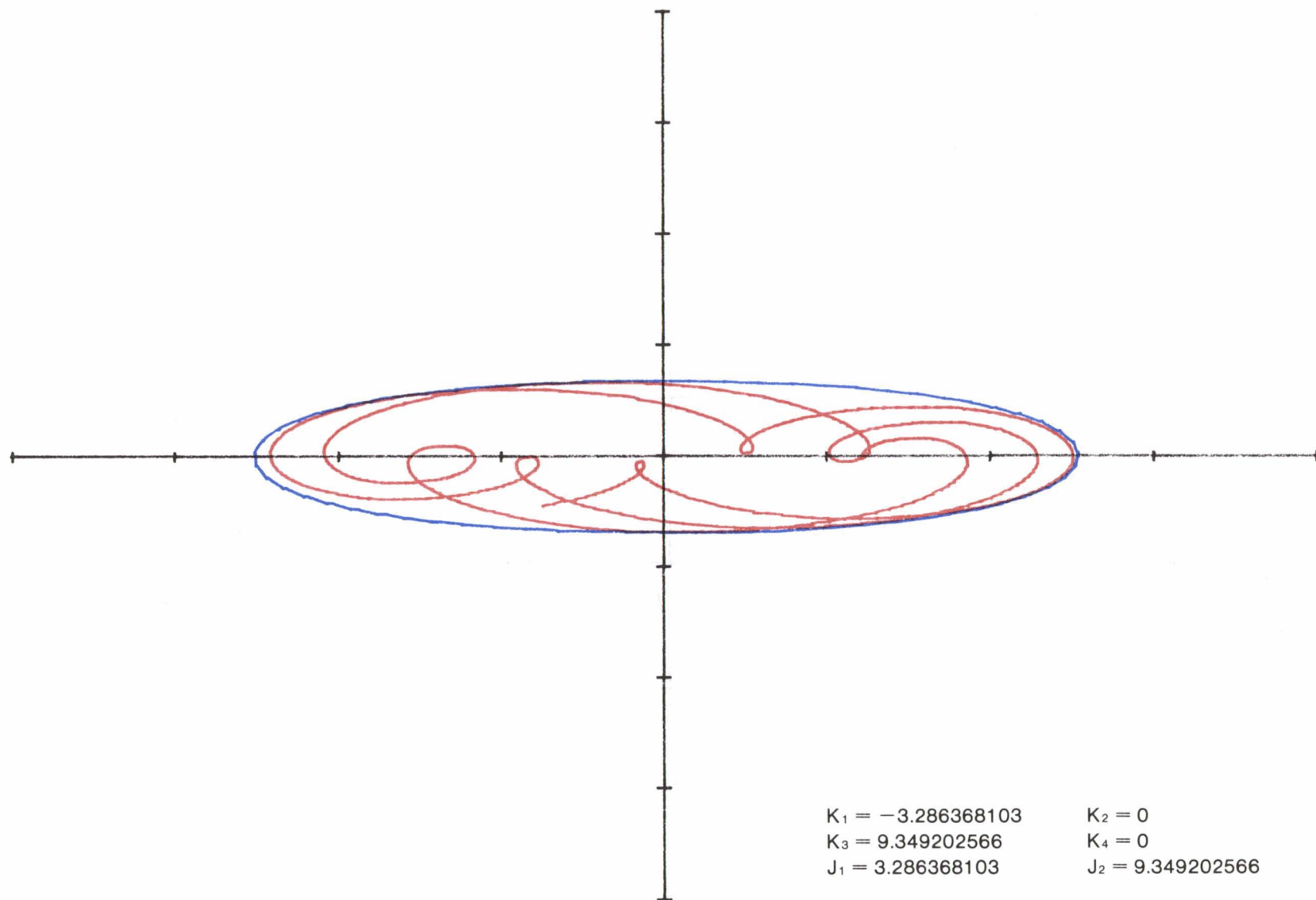
(c1) $\alpha = 1$, $\beta = 90°$, $\gamma = 0°$.
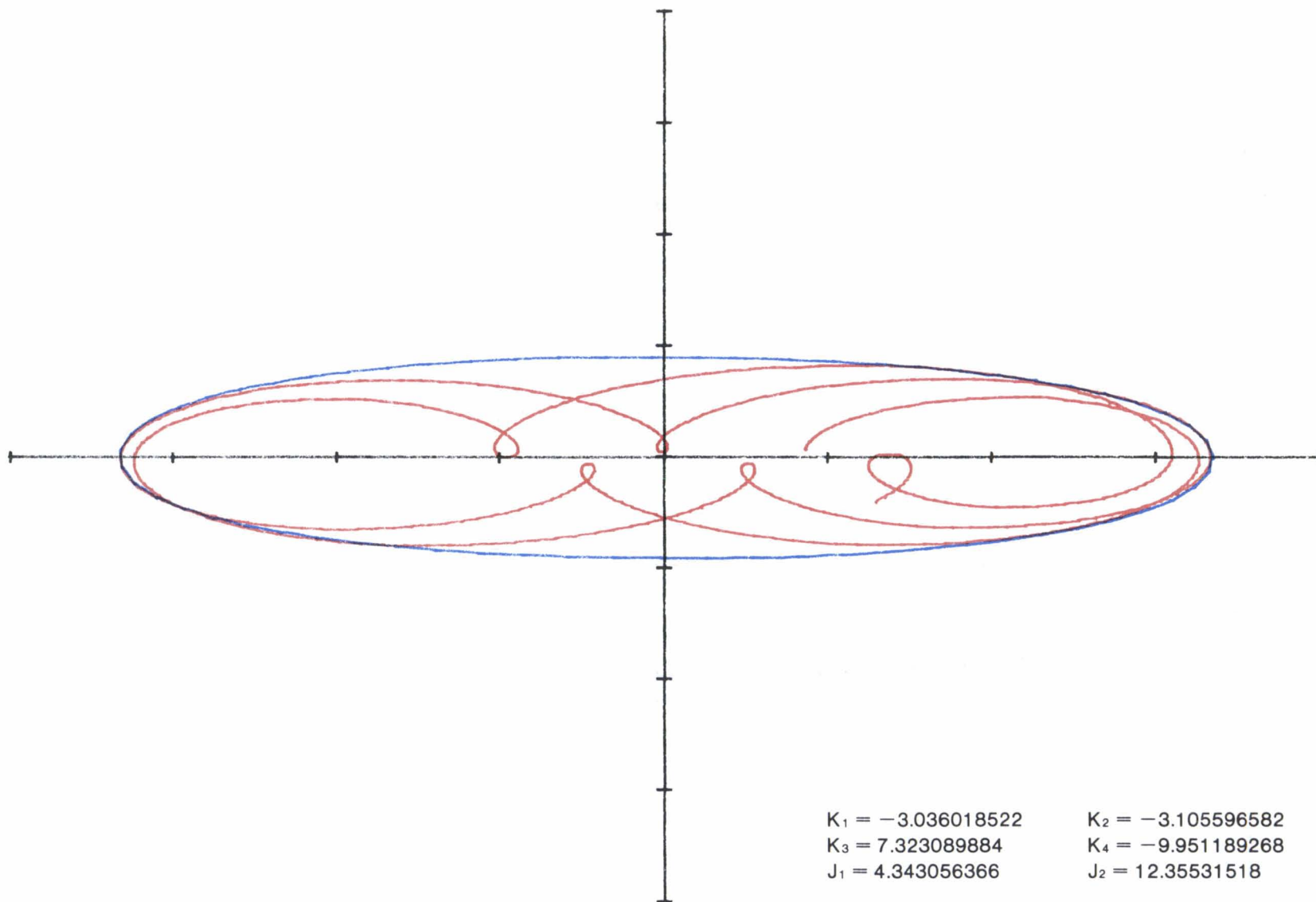
FIGURE 3.—Continued.

$K_1 = -3.036018522$ $K_2 = -3.105596582$
$K_3 = 7.323089884$ $K_4 = -9.951189268$
$J_1 = 4.343056366$ $J_2 = 12.35531518$

(c2) $\alpha = 1$, $\beta = 90°$, $\gamma = 45°$.

FIGURE 3.—Continued.

$$K_1 = -2.431621168 \qquad K_2 = -4.391976806$$
$$K_3 = 2.431621168 \qquad K_4 = 14.07310682$$
$$J_1 = 5.020183440 \qquad J_2 = 14.28163566$$

(c3) $\alpha = 1$, $\beta = 90°$, $\gamma = 90°$.

FIGURE 3.—Continued.

$K_1 = -1.827223814$ $K_2 = -3.105596582$
$K_3 = -2.459847548$ $K_4 = 9.951189268$
$J_1 = 3.603259219$ $J_2 = 10.25070816$

(c4) $\alpha = 1$, $\beta = 90°$, $\gamma = 135°$.

FIGURE 3.—Continued.

$K_1 = -1.576874233$ $K_2 = 0$
$K_3 = -4.485960231$ $K_4 = 0$
$J_1 = 1.576874233$ $J_2 = 4.485960231$

(c5) $\alpha = 1$, $\beta = 90°$, $\gamma = 180°$.

FIGURE 3.—Continued.

$$K_1 = -1.827223814 \qquad K_2 = 3.105596582$$
$$K_3 = -2.459847548 \qquad K_4 = -9.951189268$$
$$J_1 = 3.603259219 \qquad J_2 = 10.25070816$$

(c6) $\alpha = 1$, $\beta = 90°$, $\gamma = 225°$.

FIGURE 3.—Continued.

$K_1 = -2.431621168$ $K_2 = 4.391976806$
$K_3 = 2.431621168$ $K_4 = -14.07310682$
$J_1 = 5.020183440$ $J_2 = 14.28163566$

(c7) $\alpha = 1$, $\beta = 90°$, $\gamma = 270°$.

FIGURE 3.—Continued.

$K_1 = -3.036018522$ $K_2 = 3.105596582$
$K_3 = 7.323089884$ $K_4 = -9.951189268$
$J_1 = 4.343056366$ $J_2 = 12.35531518$

(c8) $\alpha = 1$, $\beta = 90°$, $\gamma = 315°$.

FIGURE 3.—Continued.

$K_1 = -2.574162752$ $K_2 = -3.497974374$
$K_3 = 8.636997215$ $K_4 = 8.834935904$
$J_1 = 4.343056366$ $J_2 = 12.35531518$

(d1) $\alpha = 1$, $\beta = 135°$, $\gamma = 0°$.

FIGURE 3.—Continued.

$$K_1 = -2.323813171 \qquad K_2 = -6.603570957$$
$$K_3 = 6.610884533 \qquad K_4 = 18.78612517$$
$$J_1 = 7.000518340 \qquad J_2 = 19.91537831$$

(d2) $\alpha = 1$, $\beta = 135°$, $\gamma = 45°$.

FIGURE 3.—Continued.

$K_1 = -1.719415817$ $K_2 = -7.889951180$
$K_3 = 1.719415817$ $K_4 = 22.90804273$
$J_1 = 8.075129744$ $J_2 = 22.97247946$

(d3) $\alpha = 1$, $\beta = 135°$, $\gamma = 90°$.

FIGURE 3.—Continued.

$K_1 = -1.115018463$ $K_2 = -6.603570957$
$K_3 = -3.172052899$ $K_4 = 18.78612517$
$J_1 = 6.697045285$ $J_2 = 19.05204500$

(d4) $\alpha = 1$, $\beta = 135°$, $\gamma = 135°$.

FIGURE 3.—Continued.

$K_1 = -0.8646688819$ $K_2 = -3.497974374$
$K_3 = -5.198165581$ $K_4 = 8.834935904$
$J_1 = 3.603259219$ $J_2 = 10.25070816$

(d5) $\alpha = 1$, $\beta = 135°$, $\gamma = 180°$.

FIGURE 3.—Continued.

$K_1 = -1.115018463$ $K_2 = -0.3923777918$
$K_3 = -3.172052899$ $K_4 = -1.116253364$
$J_1 = 1.182043660$ $J_2 = 3.362728233$

(d6) $\alpha = 1$, $\beta = 135°$, $\gamma = 225°$.

FIGURE 3.—Continued.

$K_1 = -1.719415817$ $K_2 = 0.8940024318$
$K_3 = 1.719415817$ $K_4 = -5.238170921$
$J_1 = 1.937945071$ $J_2 = 5.513150220$

(d7) $\alpha = 1$, $\beta = 135°$, $\gamma = 270°$.

FIGURE 3.—Continued.

$K_1 = -2.323813171$ $K_2 = -0.3923777918$
$K_3 = 6.610884533$ $K_4 = -1.116253364$
$J_1 = 2.356707022$ $J_2 = 6.704462386$

(d8) $\alpha = 1$, $\beta = 135°$, $\gamma = 315°$.

FIGURE 3.—Continued.

$K_1 = -0.8547469351$ $K_2 = -4.946882801$
$K_3 = 6.917581398$ $K_4 = 1.249448618$
$J_1 = 5.020183440$ $J_2 = 1.428163566$

(e1) $\alpha = 1$, $\beta = 180°$, $\gamma = 0°$.

FIGURE 3.—Continued.

$K_1 = -0.6043973540$ $K_2 = -8.052479383$
$K_3 = 4.891468716$ $K_4 = 22.44567545$
$J_1 = 8.075129744$ $J_2 = 22.97247946$

(e2) $\alpha = 1$, $\beta = 180°$, $\gamma = 45°$.

FIGURE 3.—Continued.

$K_1 = 0$ $K_2 = -9.338859607$
$K_3 = 0$ $K_4 = 26.56759300$
$J_1 = 9.338859607$ $J_2 = 26.56759300$

(e3) $\alpha = 1$, $\beta = 180°$, $\gamma = 90°$.

FIGURE 3.—Continued.

$K_1 = 6.043973540$ $K_2 = -8.052479383$
$K_3 = -4.891468716$ $K_4 = 22.44567545$
$J_1 = 8.075129744$ $J_2 = 22.97247946$

(e4) $\alpha = 1$, $\beta = 180°$, $\gamma = 135°$.

FIGURE 3.—Continued.

$K_1 = 0.8547469351$ $K_2 = -4.946882801$
$K_3 = -6.917581398$ $K_4 = 12.49448618$
$J_1 = 5.020183440$ $J_2 = 14.28163566$

(e5) $\alpha = 1$, $\beta = 180°$, $\gamma = 180°$.

FIGURE 3.—Continued.

$K_1 = 0.6043973540$ $K_2 = -1.841286218$
$K_3 = -4.891468716$ $K_4 = 2.543296910$
$J_1 = 1.937945071$ $J_2 = 5.513150222$

(e6) $\alpha = 1$, $\beta = 180°$, $\gamma = 225°$.

FIGURE 3.—Continued.

K₁ = 0 and related values shown in figure:

$K_1 = 0$ $K_2 = -0.5549059948$
$K_3 = 0$ $K_4 = -1.578620647$
$J_1 = 0.5549059948$ $J_2 = 1.578620647$

(e7) $\alpha = 1$, $\beta = 180°$, $\gamma = 270°$.

FIGURE 3.—Continued.

$K_1 = -0.6043973540$ $K_2 = -1.841286218$
$K_3 = 4.891468716$ $K_4 = 2.543296910$
$J_1 = 1.937945071$ $J_2 = 5.513150222$

(e8) $\alpha = 1$, $\beta = 180°$, $\gamma = 315°$.

FIGURE 3.—Continued.

$K_1 = 0.8646688819$ $K_2 = -3.497974374$
$K_3 = 5.198165581$ $K_4 = 8.834935904$
$J_1 = 3.603259219$ $J_2 = 10.25070816$

(f1) $\alpha = 1$, $\beta = 225°$, $\gamma = 0°$.

FIGURE 3.—Continued.

$$K_1 = 1.115018463 \qquad K_2 = -6.603570957$$
$$K_3 = 3.172052899 \qquad K_4 = 18.78612517$$
$$J_1 = 6.697045285 \qquad J_2 = 19.05204500$$

(f2) $\alpha = 1$, $\beta = 225°$, $\gamma = 45°$.

FIGURE 3.—Continued.

$K_1 = 1.719415817$ $K_2 = -7.889951180$
$K_3 = -1.719415817$ $K_4 = 22.90804273$
$J_1 = 8.075129744$ $J_2 = 22.97247946$

(f3) $\alpha = 1$, $\beta = 225°$, $\gamma = 90°$.

FIGURE 3.—Continued.

$K_1 = 2.323813171$ $K_2 = -6.603570957$
$K_3 = -6.610884533$ $K_4 = 18.78612517$
$J_1 = 7.000518340$ $J_2 = 19.91537831$

(f4) $\alpha = 1$, $\beta = 225°$, $\gamma = 135°$.

FIGURE 3.—Continued.

$K_1 = 2.574162752$ $K_2 = -3.497974374$
$K_3 = -8.636997215$ $K_4 = 8.834935904$
$J_1 = 4.343056366$ $J_2 = 12.35531518$

(f5) $\alpha = 1$, $\beta = 225°$, $\gamma = 180°$.

FIGURE 3.—Continued.

$K_1 = 2.323813171$ $K_2 = -0.3923777918$
$K_3 = -6.610884533$ $K_4 = -1.116253364$
$J_1 = 2.356707022$ $J_2 = 6.704462386$

(f6) $\alpha = 1$, $\beta = 225°$, $\gamma = 225°$.

FIGURE 3.—Continued.

$K_1 = 1.719415817$ $K_2 = 0.8940024318$
$K_3 = -1.719415817$ $K_4 = -5.238170921$
$J_1 = 1.937945071$ $J_2 = 5.513150220$

(f7) $\alpha = 1$, $\beta = 225°$, $\gamma = 270°$.

FIGURE 3.—Continued.

$$K_1 = 1.115018463 \qquad K_2 = -0.3923777918$$
$$K_3 = 3.172052899 \qquad K_4 = -1.116253364$$
$$J_1 = 1.182043360 \qquad J_2 = 3.362728233$$

(f8) $\alpha = 1$, $\beta = 225°$, $\gamma = 315°$.

FIGURE 3.—Continued.

K$_1$ = 1.576874233   K$_2$ = 0
K$_3$ = 4.485960231   K$_4$ = 0
J$_1$ = 1.576874233   J$_2$ = 4.485960231

(g1) $\alpha = 1$, $\beta = 270°$, $\gamma = 0°$.

FIGURE 3.—Continued.

$K_1 = 1.827223814$ $K_2 = -3.105596582$
$K_3 = 2.459847548$ $K_4 = 9.951189268$
$J_1 = 3.603259219$ $J_2 = 10.25070816$

(g2) $\alpha = 1$, $\beta = 270°$, $\gamma = 45°$.

FIGURE 3.—Continued.

$K_1 = 2.431621168$ $K_2 = -4.391976806$

$K_3 = -2.431621168$ $K_4 = 14.07310682$

$J_1 = 5.020183440$ $J_2 = 14.28163566$
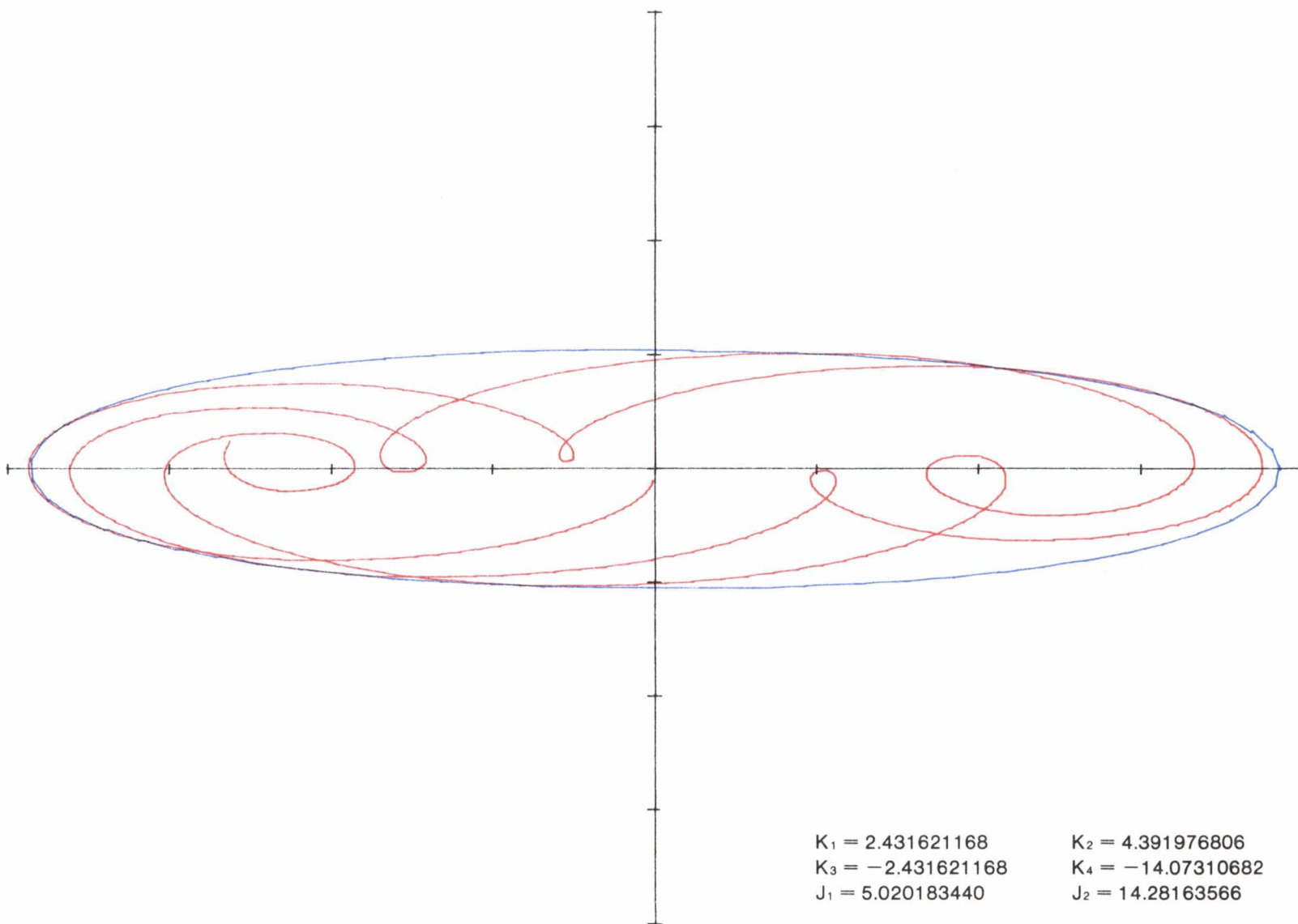
(g3) $\alpha = 1$, $\beta = 270°$, $\gamma = 90°$.

FIGURE 3.—Continued.

$K_1 = 3.036018522$ $K_2 = -3.105596582$
$K_3 = -7.323089884$ $K_4 = 9.951189268$
$J_1 = 4.343056366$ $J_2 = 12.35531518$

(g4) $\alpha = 1$, $\beta = 270°$, $\gamma = 135°$.

FIGURE 3.—Continued.

K₁ = 3.286368103   K₂ = 0
K₃ = −9.349202566   K₄ = 0
J₁ = 3.286368103   J₂ = 9.349202566

(g5) $\alpha = 1$, $\beta = 270°$, $\gamma = 180°$.

FIGURE 3.—Continued.

$K_1 = 3.036018522$ $K_2 = 3.105596582$
$K_3 = -7.323089884$ $K_4 = -9.951189268$
$J_1 = 4.343056366$ $J_2 = 12.35531518$
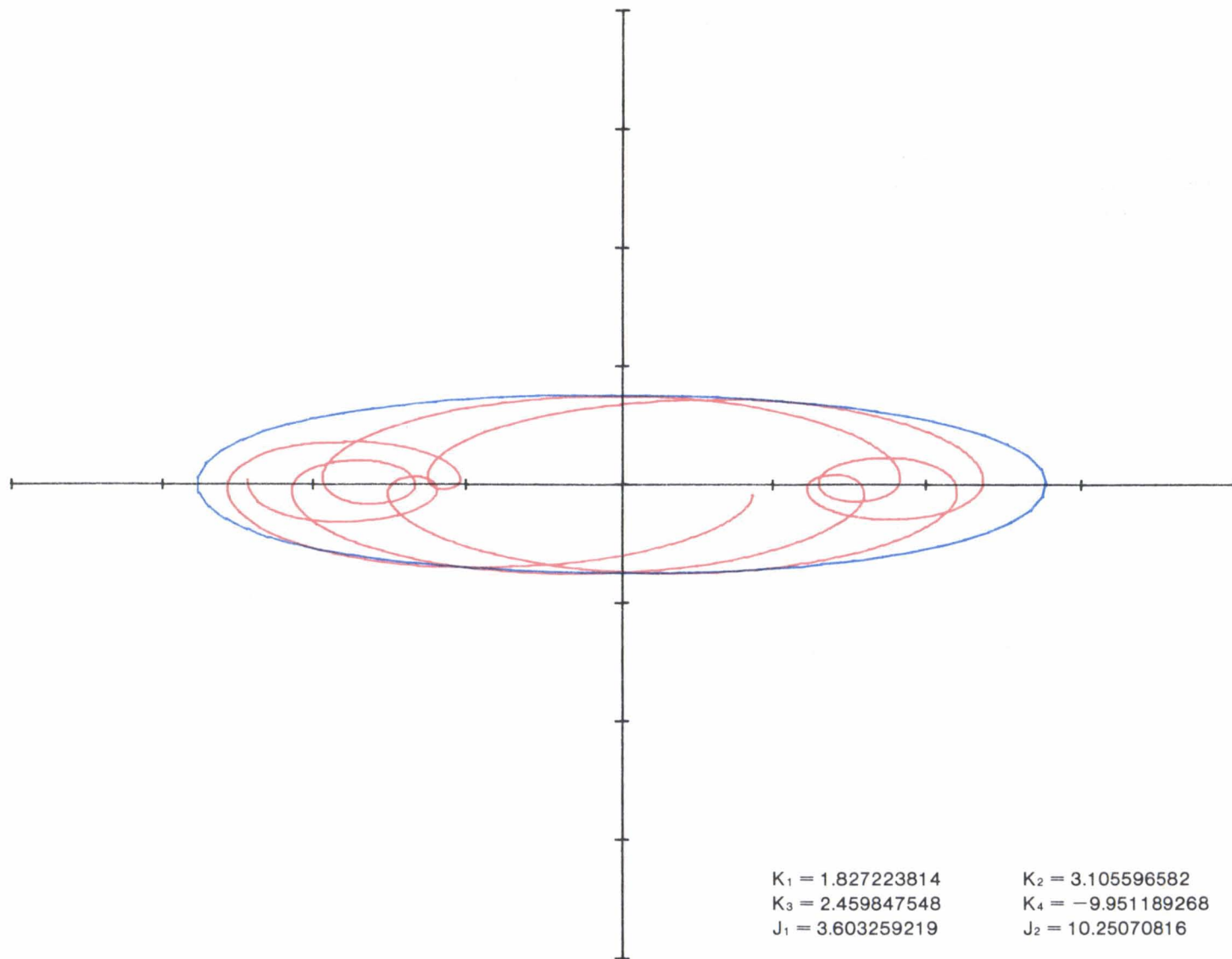
(g6) $\alpha = 1$, $\beta = 270°$, $\gamma = 225°$.

FIGURE 3.—Continued.
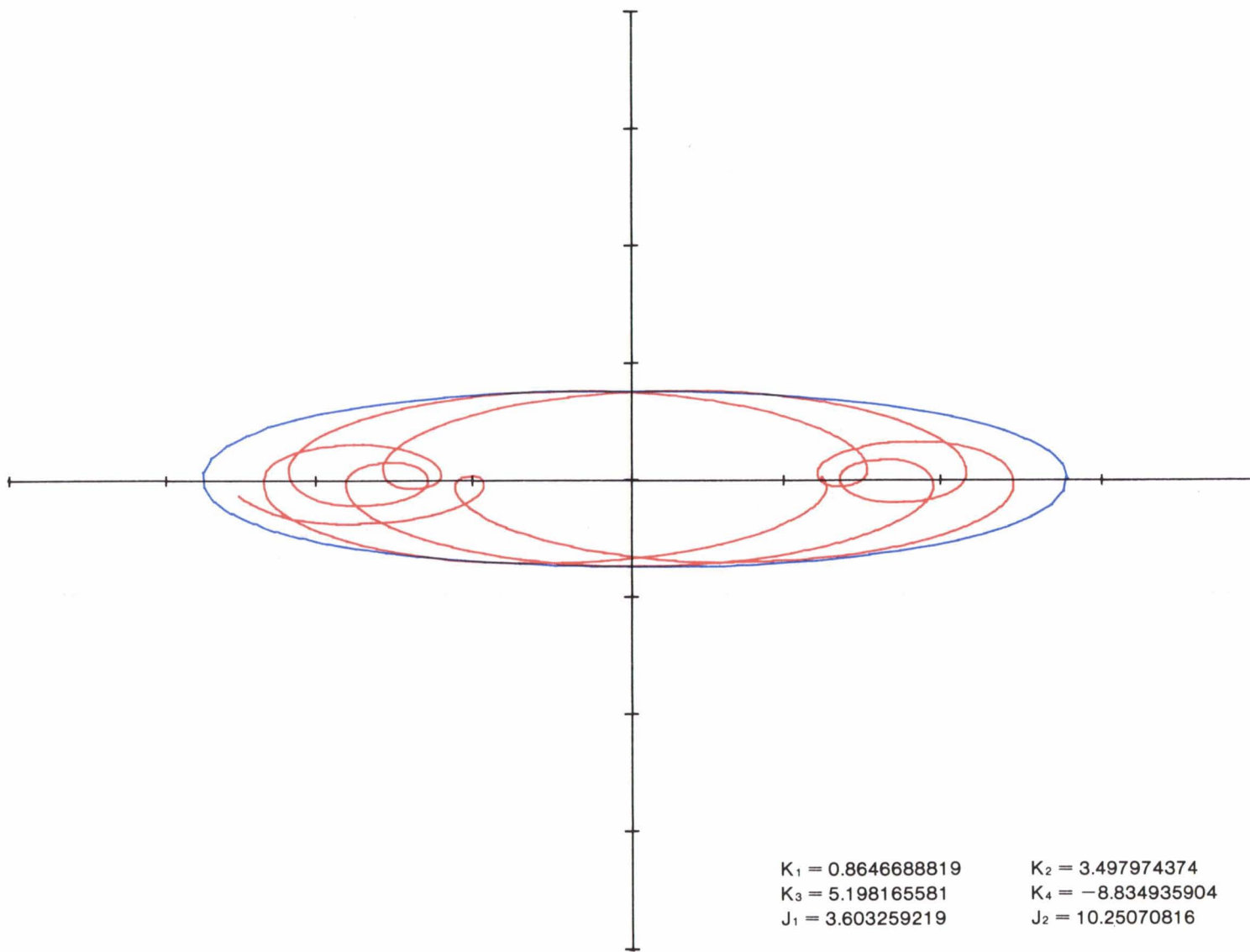
$K_1 = 2.431621168$ $K_2 = 4.391976806$
$K_3 = -2.431621168$ $K_4 = -14.07310682$
$J_1 = 5.020183440$ $J_2 = 14.28163566$

(g7) $\alpha = 1$, $\beta = 270°$, $\gamma = 270°$.

FIGURE 3.—Continued.

$K_1 = 1.827223814$ $K_2 = 3.105596582$
$K_3 = 2.459847548$ $K_4 = -9.951189268$
$J_1 = 3.603259219$ $J_2 = 10.25070816$

(g8) $\alpha = 1$, $\beta = 270°$, $\gamma = 315°$.

FIGURE 3.—Continued.

$K_1 = 0.8646688819$ $K_2 = 3.497974374$
$K_3 = 5.198165581$ $K_4 = -8.834935904$
$J_1 = 3.603259219$ $J_2 = 10.25070816$

(h1) $\alpha = 1$, $\beta = 315°$, $\gamma = 0°$.

FIGURE 3.—Continued.

$K_1 = 1.115018463$ $K_2 = 0.3923777918$
$K_3 = 3.172052899$ $K_4 = 1.116253364$
$J_1 = 1.182043360$ $J_2 = 3.362728233$

(h2) $\alpha = 1$, $\beta = 315°$, $\gamma = 45°$.

FIGURE 3.—Continued.

$K_1 = 1.719415817$ $K_2 = -0.8940024318$
$K_3 = -1.719415817$ $K_4 = 5.238170921$
$J_1 = 1.937945071$ $J_2 = 5.513150220$

(h3) $\alpha = 1$, $\beta = 315°$, $\gamma = 90°$.

FIGURE 3.—Continued.

$$K_1 = 2.323813171 \qquad K_2 = 0.3923777918$$
$$K_3 = -6.610884533 \qquad K_4 = 1.116253364$$
$$J_1 = 2.356707022 \qquad J_2 = 6.704462386$$

(h4) $\alpha = 1$, $\beta = 315°$, $\gamma = 135°$.

FIGURE 3.—Continued.

$K_1 = 2.574162752$ $K_2 = 3.497974374$
$K_3 = -8.636997215$ $K_4 = -8.834935904$
$J_1 = 4.343056366$ $J_2 = 12.35531518$

(h5) $\alpha = 1$, $\beta = 315°$, $\gamma = 180°$.

FIGURE 3.—Continued.

$K_1 = 2.323813171$ $K_2 = 6.603570957$
$K_3 = -6.610884533$ $K_4 = -18.78612517$
$J_1 = 7.000518340$ $J_2 = 19.91537831$

(h6) $\alpha = 1$, $\beta = 315°$, $\gamma = 225°$.

FIGURE 3.—Continued.

$K_1 = 1.719415817$ $K_2 = 7.889951180$
$K_3 = -1.719415817$ $K_4 = -22.90804273$
$J_1 = 8.075129744$ $J_2 = 22.97247946$

(h7) $\alpha = 1$, $\beta = 315°$, $\gamma = 270°$.

FIGURE 3.—Continued.

$K_1 = 1.115018463$ $K_2 = 6.603570957$
$K_3 = 3.172052899$ $K_4 = -18.78612517$
$J_1 = 6.697045285$ $J_2 = 19.05204500$

(h8) $\alpha = 1$, $\beta = 315°$, $\gamma = 315°$.

FIGURE 3.—Continued.

$K_1 = 0.3021986770$ $K_2 = 4.026239692$
$K_3 = -2.445734358$ $K_4 = -11.22283772$
$J_1 = 4.037564872$ $J_2 = 11.48623973$

(i1) $\alpha = 0.5$, $\beta = 0°$, $\gamma = 225°$.

FIGURE 3.—Continued.

$K_1 = 0$ $K_2 = 4.669429803$
$K_3 = 0$ $K_4 = -13.28379650$
$J_1 = 4.669429803$ $J_2 = 13.28379650$

(i2) $\alpha = 0.5$, $\beta = 0°$, $\gamma = 270°$.

FIGURE 3.—Continued.

$K_1 = -0.3021986770$ $K_2 = 4.026239692$
$K_3 = 2.445734358$ $K_4 = -11.22283772$
$J_1 = 4.037564872$ $J_2 = 11.48623973$

(i3) $\alpha = 0.5$, $\beta = 0°$, $\gamma = 315°$.

FIGURE 3.—Continued.

$K_1 = -0.5575092315$ $K_2 = 3.301785478$
$K_3 = -1.586026450$ $K_4 = -9.393062586$
$J_1 = 3.348522643$ $J_2 = 9.526022499$

(j1) $\alpha = 0.5$, $\beta = 45°$, $\gamma = 225°$.

FIGURE 3.—Continued.

$K_1 = -0.8597079085$ $K_2 = 3.944975590$
$K_3 = 0.8597079085$ $K_4 = -11.45402136$
$J_1 = 4.037564872$ $J_2 = 11.48623973$

(j2) $\alpha = 0.5$, $\beta = 45°$, $\gamma = 270°$.

FIGURE 3.—Continued.

$K_1 = -1.161906586 \qquad K_2 = 3.301785478$
$K_3 = 3.305442267 \qquad K_4 = -9.393062586$
$J_2 = 3.500259170 \qquad J_2 = 9.957689156$

(j3) $\alpha = 0.5$, $\beta = 45°$, $\gamma = 315°$.

FIGURE 3.—Continued.

$K_1 = -1.161906586 \quad K_2 = -3.301785478$
$K_3 = 3.305442267 \quad K_4 = 9.393062586$
$J_1 = 3.500259170 \quad J_2 = 9.957689156$

(k1) $\alpha = 0.5$, $\beta = 135°$, $\gamma = 45°$.

FIGURE 3.—Continued.

$K_1 = -0.8597079085$ $K_2 = -3.944975590$
$K_3 = 0.8597079085$ $K_4 = 11.45402136$
$J_1 = 4.037564872$ $J_2 = 11.48623973$

(k2) $\alpha = 0.5$, $\beta = 135°$, $\gamma = 90°$.

FIGURE 3.—Continued.

$K_1 = -0.5575092315$ $K_2 = -3.301785478$
$K_3 = -1.586026450$ $K_4 = 9.393062586$
$J_1 = 3.348522643$ $J_2 = 9.526022499$

(k3) $\alpha = 0.5$, $\beta = 135°$, $\gamma = 135°$.

FIGURE 3.—Continued.

$K_1 = -0.3021986770 \qquad K_2 = -4.026239692$
$K_3 = 2.445734358 \qquad K_4 = 11.22283772$
$J_1 = 4.037564872 \qquad J_2 = 11.48623973$

(ll) $\alpha = 0.5$, $\beta = 180°$, $\gamma = 45°$.

FIGURE 3.—Continued.

$K_1 = 0$ $K_2 = -4.669429803$
$K_3 = 0$ $K_4 = 13.28379650$
$J_1 = 4.669429803$ $J_2 = 13.28379650$

(12) $\alpha = 0.5$, $\beta = 180°$, $\gamma = 90°$.

FIGURE 3.—Continued.

$K_1 = 0.3021986770$ $K_2 = -4.026239692$
$K_3 = -2.445734358$ $K_4 = 11.22283772$
$J_1 = 4.037564872$ $J_2 = 11.48623973$

(13) $\alpha = 0.5$, $\beta = 180°$, $\gamma = 135°$.

FIGURE 3.—Continued.

$K_1 = 0.5575092315$ $K_2 = -3.301785478$
$K_3 = 1.586026450$ $K_4 = 9.393062586$
$J_1 = 3.348522643$ $J_2 = 9.526022499$

(m1) $\alpha = 0.5$, $\beta = 225°$, $\gamma = 45°$.

FIGURE 3.—Continued.

$K_1 = 0.8597079085$ $K_2 = -3.944975590$
$K_3 = -0.8597079085$ $K_4 = 11.45402136$
$J_1 = 4.037564872$ $J_2 = 11.48623973$

(m2) $\alpha = 0.5$, $\beta = 225°$, $\gamma = 90°$.

FIGURE 3.—Continued.

$K_1 = 1.161906586$ $K_2 = -3.301785478$
$K_3 = -3.305442267$ $K_4 = 9.393062586$
$J_1 = 3.500259170$ $J_2 = 9.957689156$

(m3) $\alpha = 0.5$, $\beta = 225°$, $\gamma = 135°$.

FIGURE 3.—Continued.

$K_1 = 1.161906586$ $K_2 = 3.301785478$
$K_3 = -3.305442267$ $K_4 = -9.393062586$
$J_1 = 3.500259170$ $J_2 = 9.957689156$

(n1) $\alpha = 0.5$, $\beta = 315°$, $\gamma = 225°$.

FIGURE 3.—Continued.

$K_1 = 0.8597079085$ $K_2 = 3.944975590$
$K_3 = -0.8597079085$ $K_4 = -11.45402136$
$J_1 = 4.037564872$ $J_2 = 11.48623973$

(n2) $\alpha = 0.5$, $\beta = 315°$, $\gamma = 270°$.

FIGURE 3.—Continued.

$K_1 = 0.5575092315$ $K_2 = 3.301785478$
$K_3 = 1.586026450$ $K_4 = -9.393062586$
$J_1 = 3.348522643$ $J_2 = 9.526022499$

(n3) $\alpha = 0.5$, $\beta = 315°$, $\gamma = 315°$.

FIGURE 3.—Concluded.

$K_1 = 0.8547469351$ $K_2 = -4.946882801$
$K_3 = -6.917581398$ $K_4 = 12.49448618$
$J_1 = 5.020183440$ $J_2 = 14.28163566$
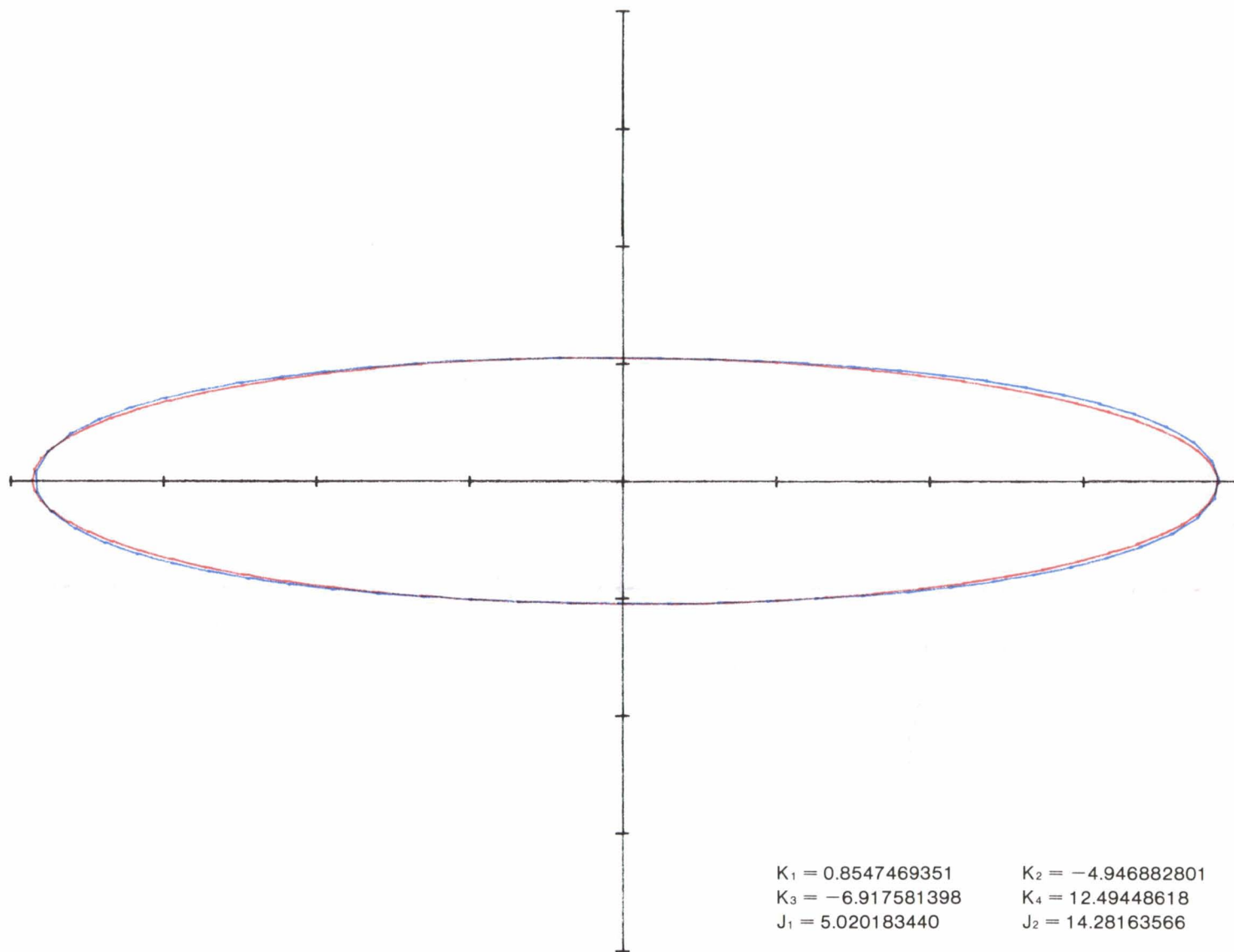
(a) $\alpha = 1$, $\beta = 180°$, $\gamma = 180°$.

FIGURE 4.—Comparison of approximate and exact bounds.

$K_1 = 0$ $K_2 = 9.338859607$
$K_3 = 0$ $K_4 = -26.56759300$
$J_1 = 9.338859607$ $J_2 = 26.56759300$

(b) $\alpha = 1$, $\beta = 0°$, $\gamma = 270°$.

FIGURE 4.—Continued.

$K_1 = 0$ $K_2 = 4.669429803$
$K_3 = 0$ $K_4 = -13.28379650$
$J_1 = 4.669429803$ $J_2 = 13.28379650$

(c) $\alpha = 0.5$, $\beta = 0°$, $\gamma = 270°$.

FIGURE 4.—Continued.

$$K_1 = -2.431621168 \qquad K_2 = 0$$
$$K_3 = 2.431621168 \qquad K_4 = 0$$
$$J_1 = 2.431621168 \qquad J_2 = 2.431621168$$

(d) $\alpha = 0$, $\beta = 0$, $\gamma = 90°$.

FIGURE 4.—Continued.

$K_1 = -1.719415817$ $K_2 = 3.497974374$
$K_3 = 1.719415817$ $K_4 = -8.834935904$
$J_1 = 3.897719266$ $J_2 = 9.000693483$

(e) $\alpha = 0$, $\beta = 0$, $\gamma = 45°$.

FIGURE 4.—Continued.

$$K_1 = 0 \qquad\qquad K_2 = 4.946882801$$
$$K_3 = 0 \qquad\qquad K_4 = -12.49448618$$
$$J_1 = 4.946882801 \qquad J_2 = 12.49448618$$

(f) $\alpha = 0$, $\beta = 0$, $\gamma = 0°$.

FIGURE 4.—Continued.

$K_1 = -0.8547469351$ $K_2 = 0$
$K_3 = 6.917581398$ $K_4 = 0$
$J_1 = 0.8547469351$ $J_2 = 6.917581398$

(g) $\alpha = 0, \beta = 0, \gamma = 0°$.

FIGURE 4.—Continued.

$$K_1 = -0.6043973540 \qquad K_2 = -3.105596582$$
$$K_3 = 4.891468716 \qquad K_4 = 9.951189268$$
$$J_1 = 3.163862559 \qquad J_2 = 11.08840088$$

(h) $\alpha = 0$, $\beta = 0$, $\gamma = 45°$.

FIGURE 4.—Continued.

$K_1 = 0$ $K_2 = -4.391976806$
$K_3 = 0$ $K_4 = 14.07310682$
$J_1 = 4.391976806$ $J_2 = 14.07310682$

(i) $\alpha = 0$, $\beta = 0$, $\gamma = 90°$.

FIGURE 4.—Continued.

$K_1 = 0$ $K_2 = 4.391976806$
$K_3 = 0$ $K_4 = -14.07310682$
$J_1 = 4.391976806$ $J_2 = 14.07310682$

(j) $\alpha = 0,\ \beta = 0,\ \gamma = 270°$.

FIGURE 4.—Concluded.

K₁ = 0   K₂ = 4.669429803
K₃ = 0   K₄ = −13.28379650
J₁ = 4.669429803   J₂ = 13.28379650
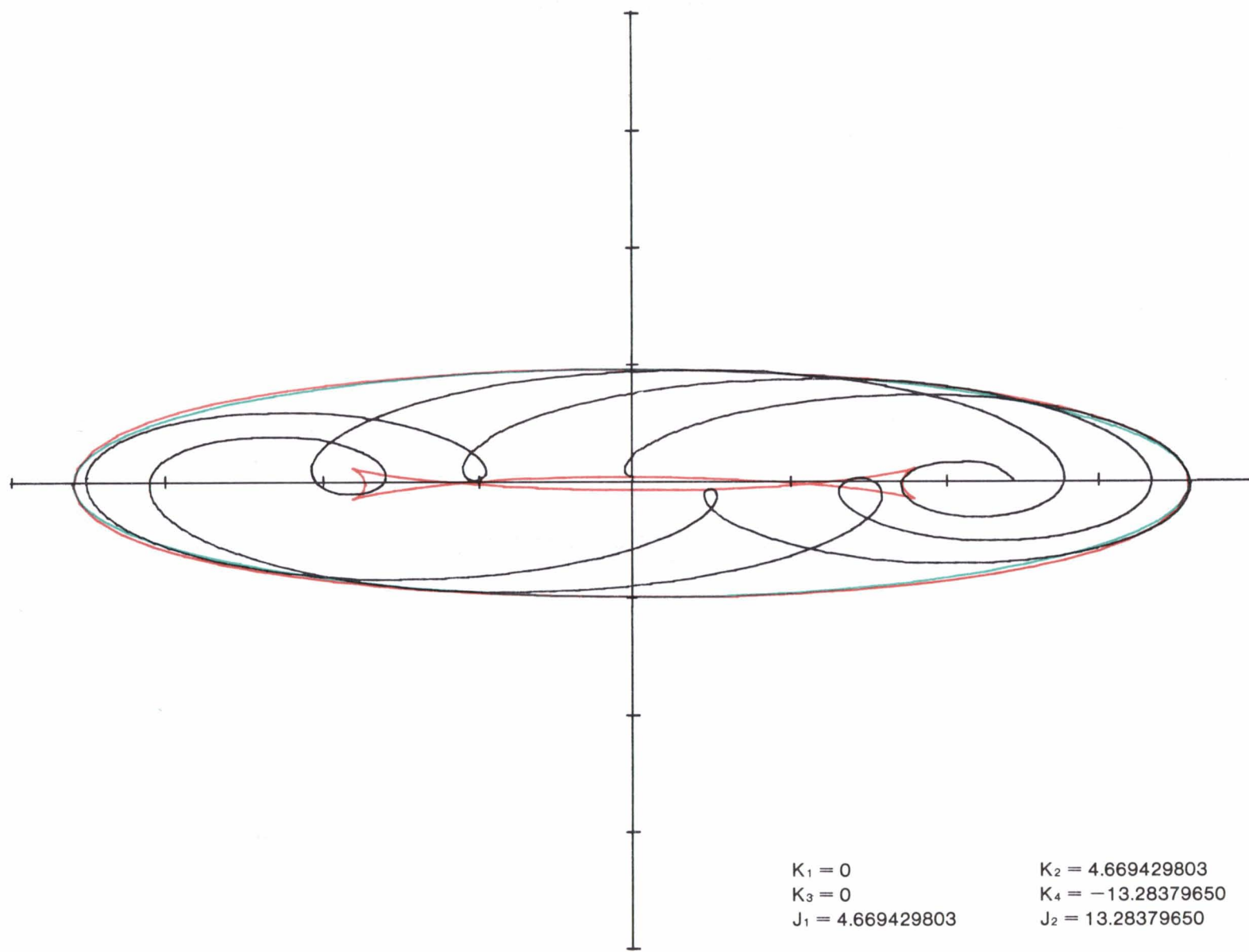
FIGURE 5.—Typical inner bounding curve showing excluded region ($\alpha = 0.5$, $\beta = 0$, $\gamma = 270°$).

$K_1 = 0$ $K_2 = 4.669429803$

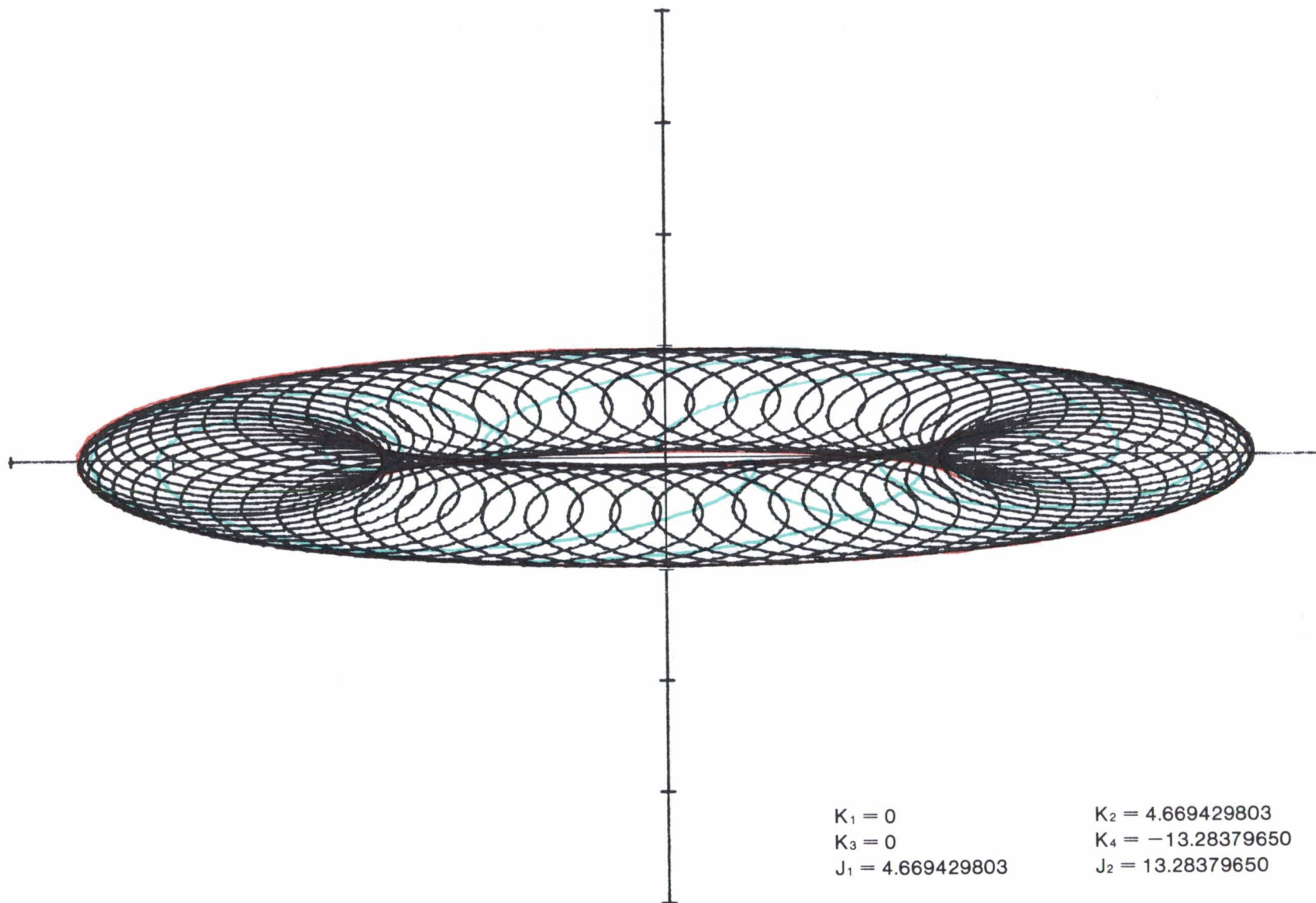$K_3 = 0$ $K_4 = -13.28379650$

$J_1 = 4.669429803$ $J_2 = 13.28379650$

FIGURE 6.—Family of smaller ellipses ($\alpha = 0.5$, $\beta = 0$, $\gamma = 270°$).

$K_1 = -1.719415817$ $K_2 = 3.497974374$
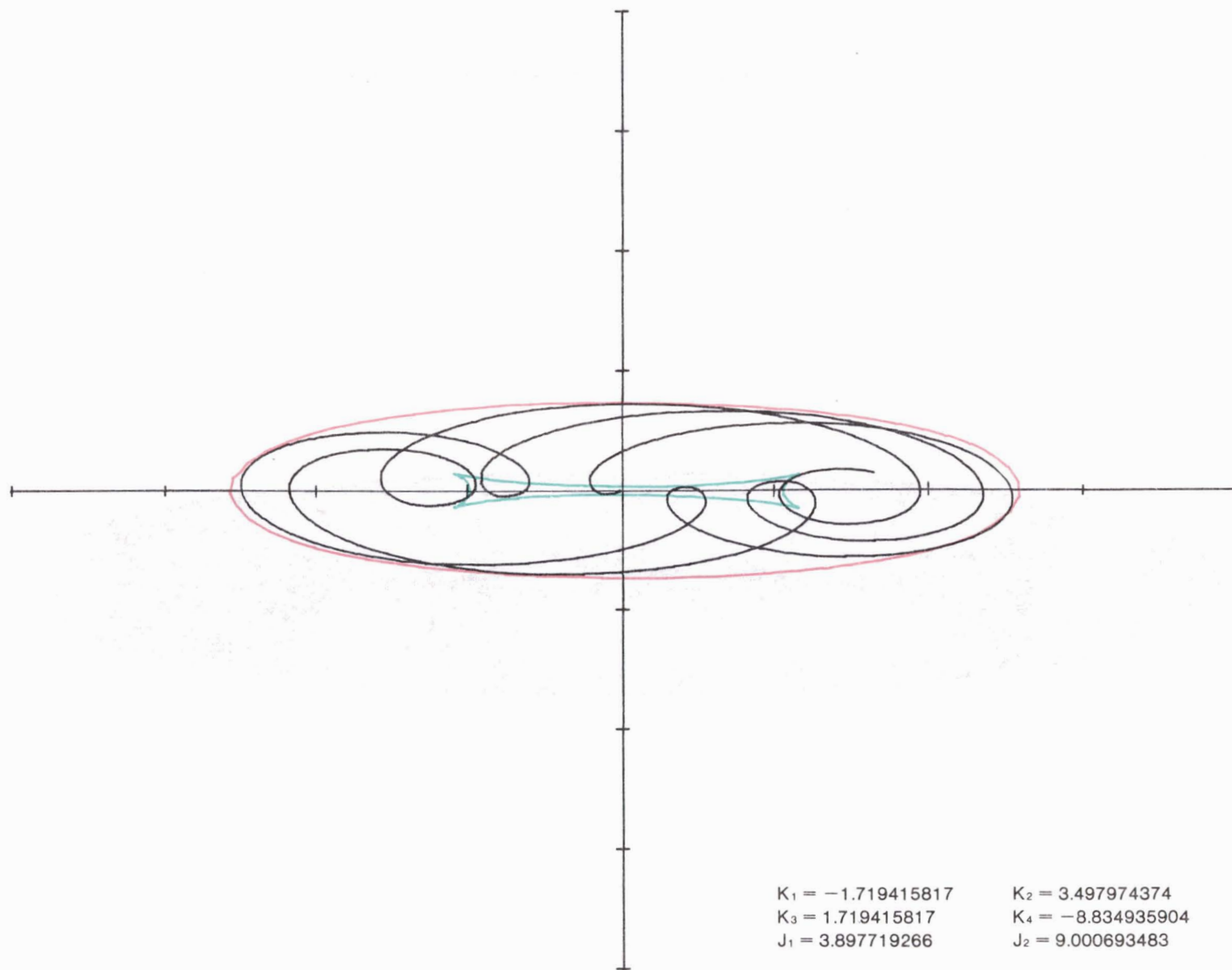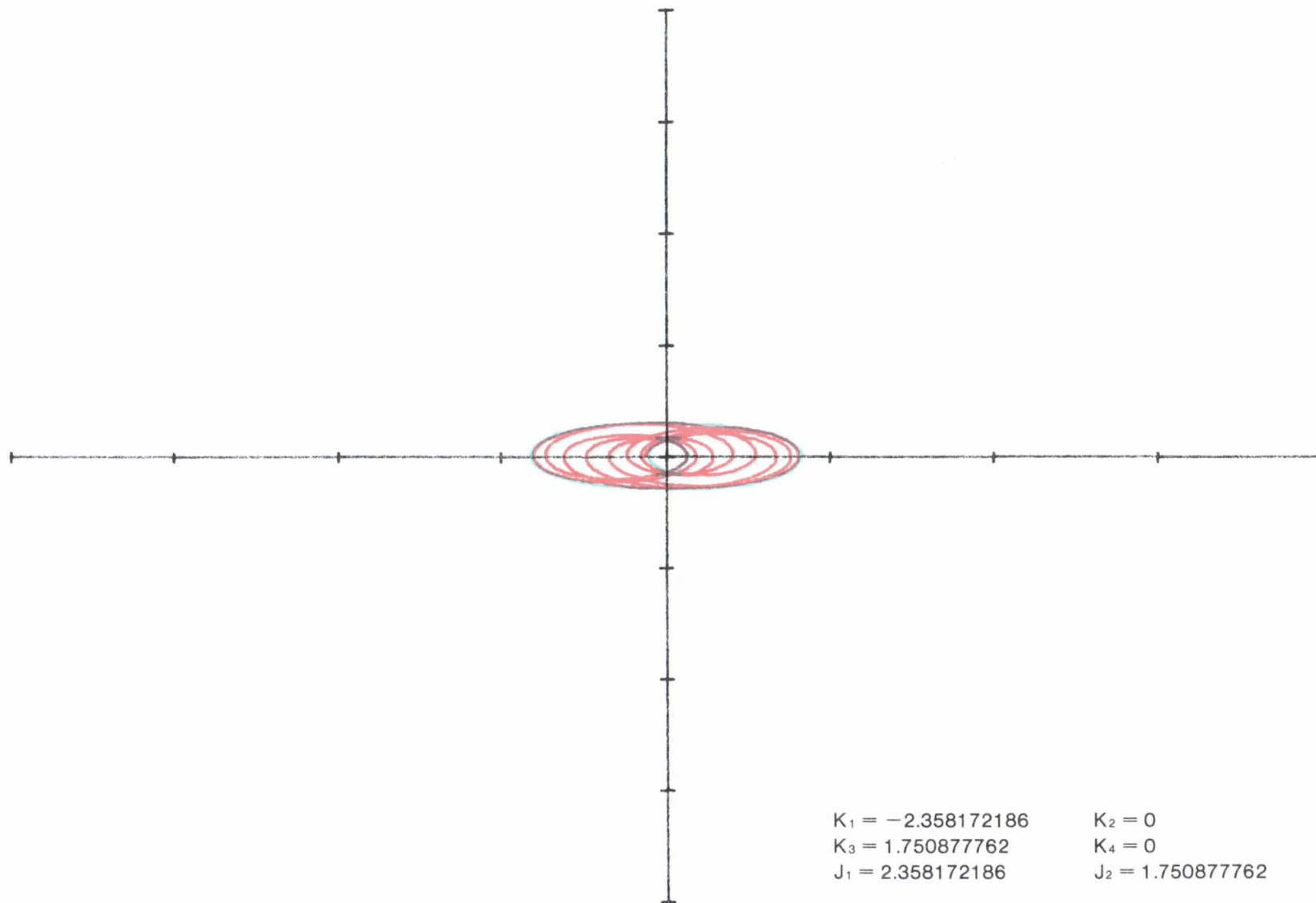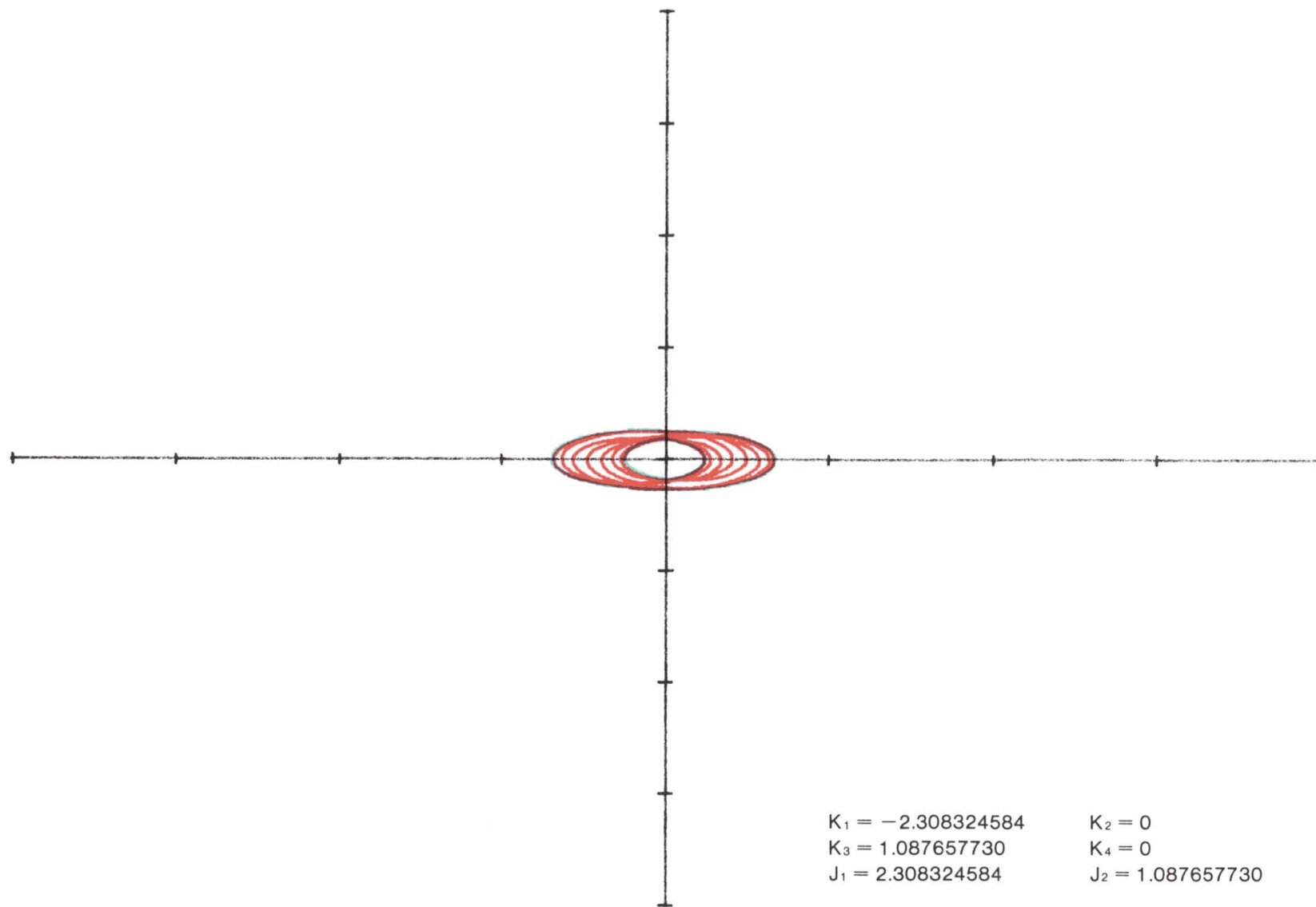$K_3 = 1.719415817$ $K_4 = -8.834935904$
$J_1 = 3.897719266$ $J_2 = 9.000693483$

FIGURE 7.—An example for which there is no inner excluded region ($\alpha = 0$, $\beta = 0$, $\gamma = 45°$).

$$K_1 = -2.358172186 \qquad K_2 = 0$$
$$K_3 = 1.750877762 \qquad K_4 = 0$$
$$J_1 = 2.358172186 \qquad J_2 = 1.750877762$$

(a) $\alpha = 0$, $\beta = -0.1$, $\gamma = 90°$.

FIGURE 8.—Trajectories for which $J_1 > J_2$.
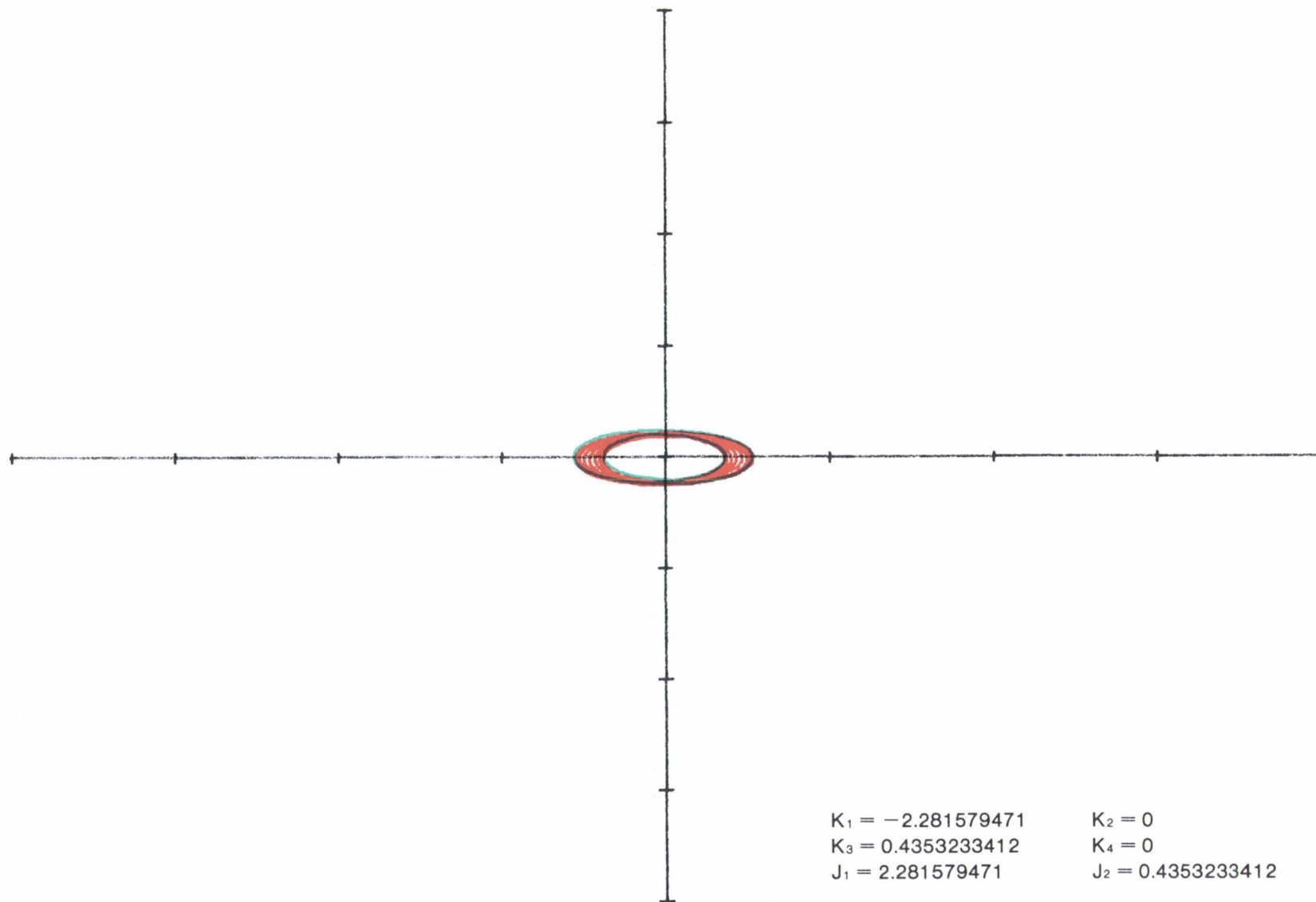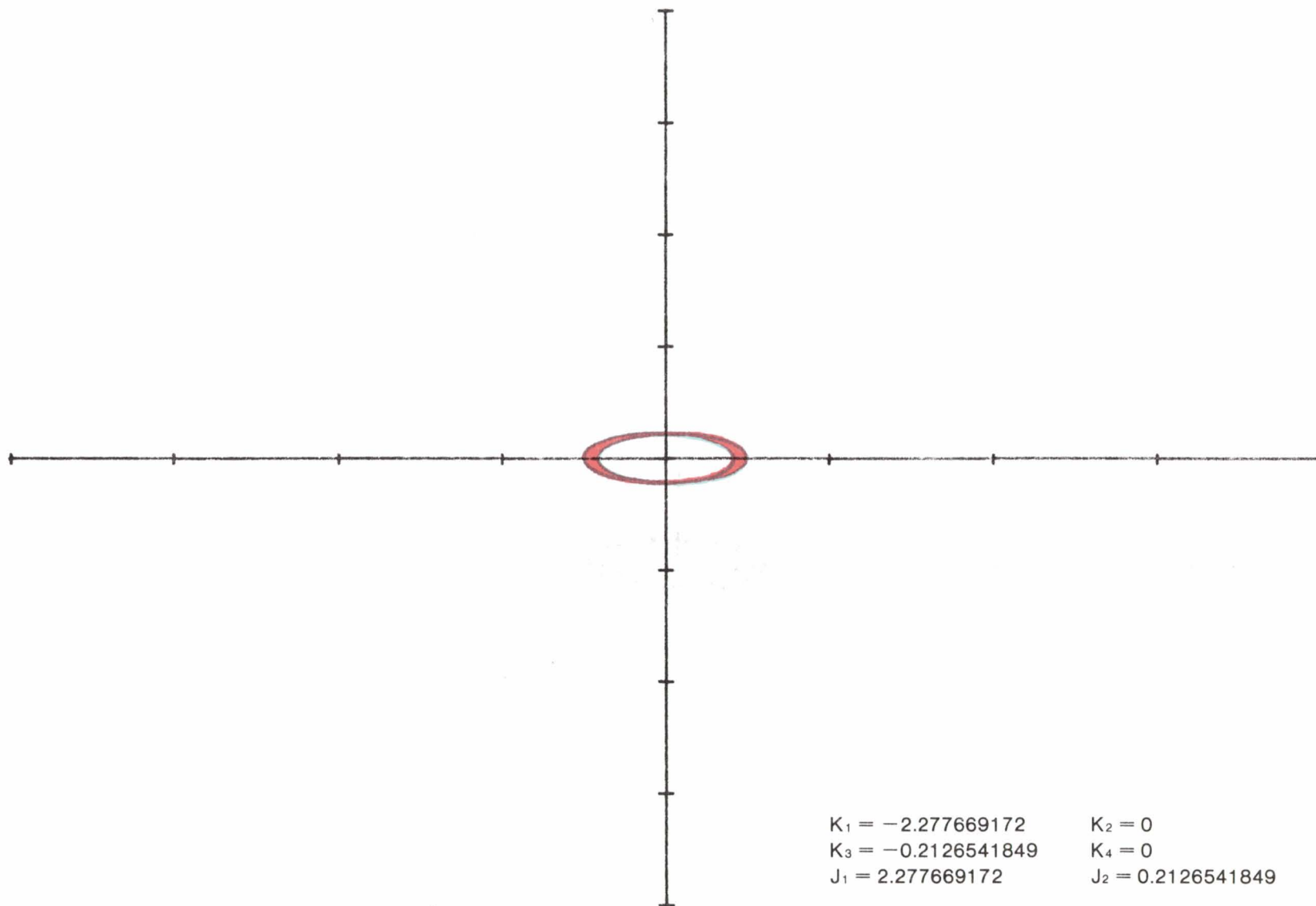
$K_1 = -2.308324584$ $K_2 = 0$
$K_3 = 1.087657730$ $K_4 = 0$
$J_1 = 2.308324584$ $J_2 = 1.087657730$

(b) $\alpha = 0$, $\beta = -0.2$, $\gamma = 90°$.

FIGURE 8.—Continued.

$K_1 = -2.281579471$ $K_2 = 0$
$K_3 = 0.4353233412$ $K_4 = 0$
$J_1 = 2.281579471$ $J_2 = 0.4353233412$

(c) $\alpha = 0$, $\beta = -0.3$, $\gamma = 90°$.

FIGURE 8.—Continued.

$K_1 = -2.277669172$ $K_2 = 0$
$K_3 = -0.2126541849$ $K_4 = 0$
$J_1 = 2.277669172$ $J_2 = 0.2126541849$

(d) $\alpha = 0$, $\beta = -0.4$, $\gamma = 90°$.

FIGURE 8.—Continued.

$$K_1 = -2.296554551 \qquad K_2 = 0$$
$$K_3 = -0.8627600257 \qquad K_4 = 0$$
$$J_1 = 2.296554551 \qquad J_2 = 0.8627600257$$

(e) $\alpha = 0$, $\beta = -0.5$, $\gamma = 90°$.

FIGURE 8.—Continued.

$K_1 = -2.338424620$ $K_2 = 0$
$K_3 = -1.521500659$ $K_4 = 0$
$J_1 = 2.338424620$ $J_2 = 1.521500659$

(f) $\alpha = 0$, $\beta = -0.6$, $\gamma = 90°$.

FIGURE 8.—Continued.

$K_1 = -2.403698429$ $K_2 = 0$
$K_3 = -2.195468982$ $K_4 = 0$
$J_1 = 2.403698429$ $J_2 = 2.195468982$
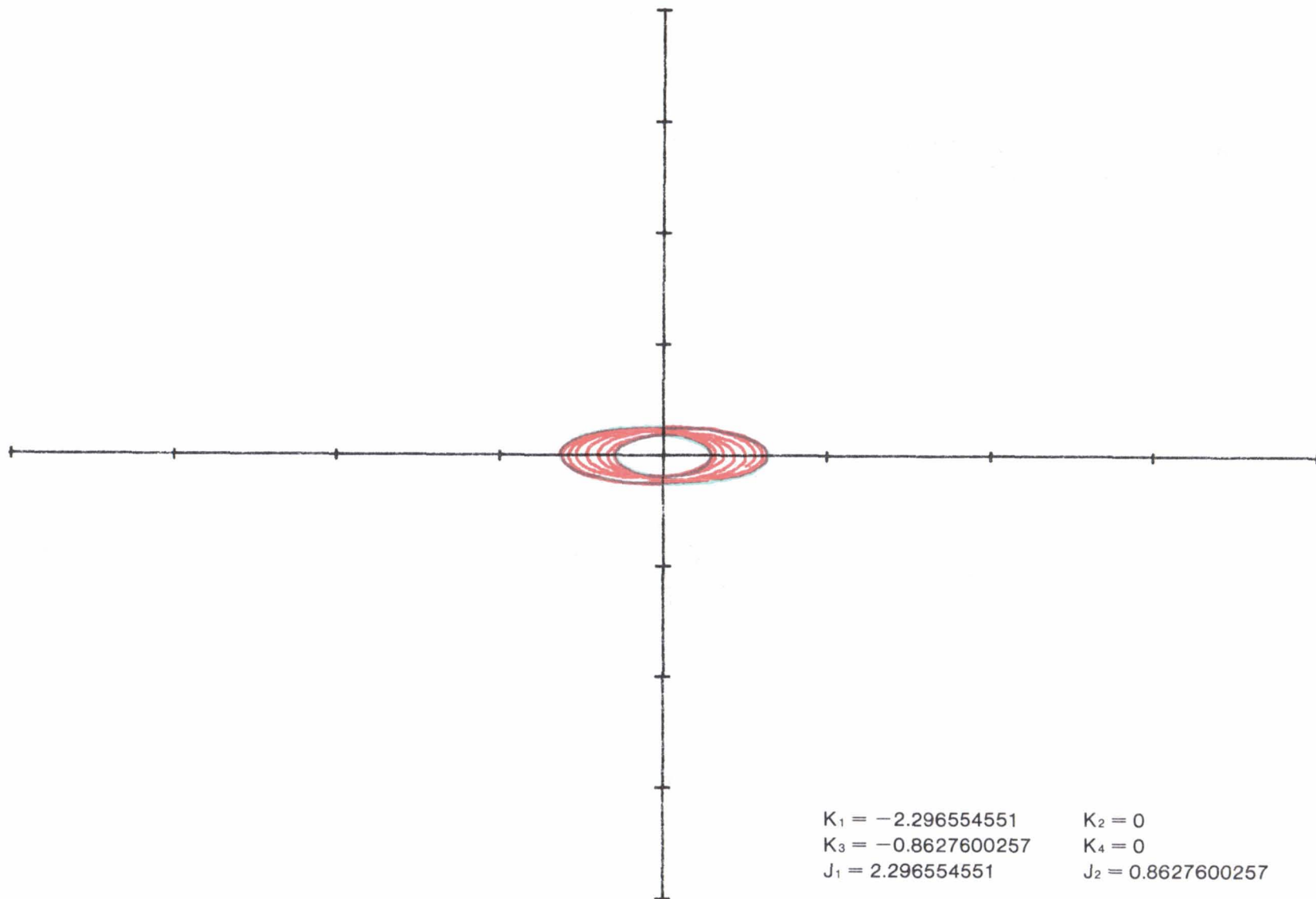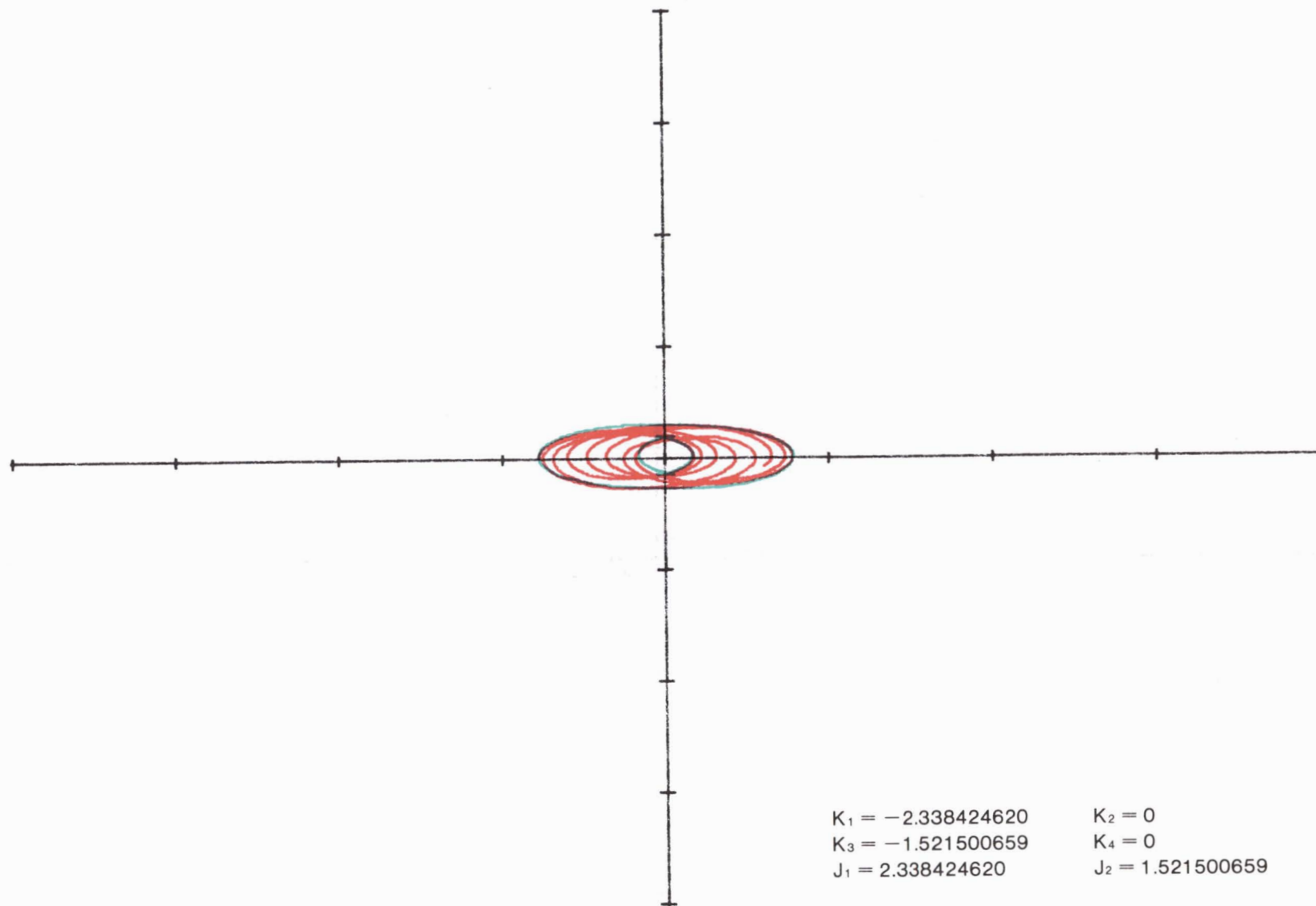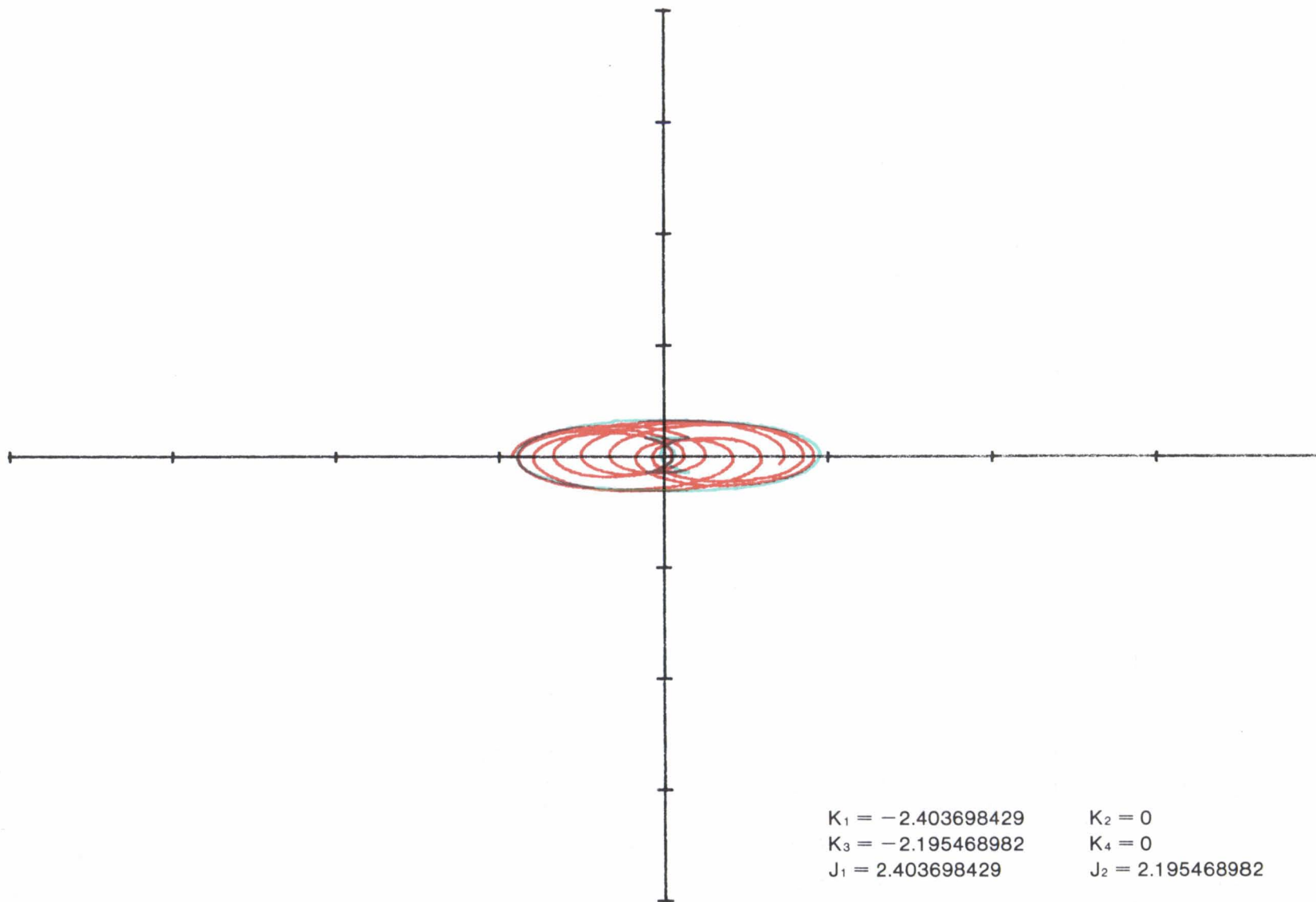
(g) $\alpha = 0$, $\beta = -0.7$, $\gamma = 90°$.

FIGURE 8.—Concluded.

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15. Supplementary Notes

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16. Abstract

The motion of a point particle in the neighborhood of a triangular libration point (L_4 or L_5) in the linearized, restricted problem of three bodies in the plane is described. The derivation of the equations of motion is standard. From these equations, three invariants of the motion are obtained; the Jacobi integral is expressed linearly in terms of two of these. The trajectories for varied initial conditions are drawn, and a complete geometric description of the particle motion is given in elementary terms. Each trajectory has an exterior boundary curve; its equation is found. An approximation to this boundary curve had been known; the two curves are compared graphically. For certain initial conditions, there is an interior region from which the trajectory is excluded; the equation of the boundary of this region is given.

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